

## TEICHMÜLLER FLOW AND WEIL-PETERSSON FLOW

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ABSTRACT. For an oriented surface  $S$  of genus  $g \geq 0$  with  $m \geq 0$  punctures and  $3g - 3 + m \geq 2$ , let  $\mathcal{Q}(S)$  and  $\mathcal{Q}_{WP}(S)$  be the moduli space of area one quadratic differentials and of quadratic differentials of unit norm for the Weil-Petersson metric, respectively. We show that there is a Borel subset  $\mathcal{E}$  of  $\mathcal{Q}(S)$  which is invariant under the Teichmüller flow  $\Phi_{\mathcal{T}}^t$  and of full measure for every invariant Borel probability measure, and there is a measurable map  $\Lambda : \mathcal{E} \rightarrow \mathcal{Q}_{WP}(S)$  which conjugates  $\Phi_{\mathcal{T}}^t|_{\mathcal{E}}$  into the Weil-Petersson flow  $\Phi_{WP}^t$ . This conjugacy induces a continuous injection of the space of all  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures on  $\mathcal{Q}(S)$  into the space of all  $\Phi_{WP}^t$ -invariant Borel probability measures on  $\mathcal{Q}_{WP}(S)$ . The map  $\Theta$  is not surjective, but its image contains the Lebesgue Liouville measure. A measure not in the image corresponds to a locally finite infinite invariant Borel measure on  $\mathcal{Q}(S)$ .

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## 1. INTRODUCTION

An oriented surface  $S$  of finite type is a closed surface of genus  $g \geq 0$  from which  $m \geq 0$  points, so-called *punctures*, have been deleted. We assume that  $3g - 3 + m \geq 2$ , i.e. that  $S$  is not a sphere with at most four punctures or a torus with at most one puncture. We then call the surface  $S$  *nonexceptional*.

Since the Euler characteristic of  $S$  is negative, the *Teichmüller space*  $\mathcal{T}(S)$  of  $S$  is the quotient of the space of all complete finite volume hyperbolic metrics on  $S$  under the action of the group of diffeomorphisms of  $S$  which are isotopic to the identity. The *mapping class group*  $\text{Mod}(S)$  of all isotopy classes of orientation preserving diffeomorphisms of  $S$  acts properly discontinuously on  $\mathcal{T}(S)$ .

There are two  $\text{Mod}(S)$ -invariant metrics on Teichmüller space  $\mathcal{T}(S)$  which have been studied extensively in the past: The so-called *Teichmüller metric* and the *Weil-Petersson metric*.

The *Teichmüller metric* is a complete  $\text{Mod}(S)$ -invariant Finsler metric on  $\mathcal{T}(S)$ . It is customary to view this metric as a metric on the cotangent bundle of Teichmüller space. This cotangent bundle is the bundle of holomorphic quadratic differentials over  $\mathcal{T}(S)$ . The unit sphere bundle of the metric is the bundle  $\tilde{\mathcal{Q}}(S)$  of quadratic differentials of area one. Although the large scale geometry of the Teichmüller metric does not resemble a metric of non-positive curvature, any two points of  $\mathcal{T}(S)$  can be connected by a unique geodesic. We call such a geodesic a *Teichmüller geodesic*.

Teichmüller distances can be effectively estimated (see [R07b]), and there are many recent results on the asymptotic behavior of Teichmüller geodesics. We refer to [LM10, LR11] for more information.

The Teichmüller metric defines a *geodesic flow* on  $\tilde{\mathcal{Q}}(S)$  which is equivariant with respect to the action of the mapping class group and hence projects to a flow  $\Phi_{\mathcal{T}}^t$  on the *moduli space*

$$\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\text{Mod}(S)$$

of area one quadratic differentials.

Although  $\mathcal{Q}(S)$  is not compact, it admits many  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures. For example, there are countably many periodic orbits for the Teichmüller flow on  $\mathcal{Q}(S)$ , and each such orbit supports a natural invariant Borel probability measure. These periodic orbits are in bijection with conjugacy classes of pseudo-Anosov mapping classes, and they can be counted according to their lengths [EM11, H13]. Borel probability measures supported on periodic orbits are dense in the space  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  of all  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures on  $\mathcal{Q}(S)$  equipped with the weak\*-topology. However, for every compact set  $K \subset \mathcal{Q}(S)$  there are periodic orbits which do not intersect  $K$  [H05] and hence the space  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  is non-compact: There are sequences of such measures supported on periodic orbits which converge weakly to the trivial measure of vanishing total mass.

The *Weil-Petersson metric* is a  $\text{Mod}(S)$ -invariant Kähler metric on  $\mathcal{T}(S)$  of negative sectional curvature which induces an incomplete distance  $d_{WP}$ . The completion  $\overline{\mathcal{T}(S)}$  of  $\mathcal{T}(S)$  with respect to  $d_{WP}$  is a  $\text{Cat}(0)$ -space, however it is not locally compact. As a consequence, any two points in  $\overline{\mathcal{T}(S)}$  can be connected by a unique geodesic. Such a geodesic will be called a *Weil-Petersson geodesic* in the sequel. Finite length Weil-Petersson geodesic arcs with both endpoints in  $\mathcal{T}(S)$  are entirely contained in  $\mathcal{T}(S)$  (Corollary 5.4 of [W87]).

The Weil-Petersson metric can be viewed as a  $\text{Mod}(S)$ -invariant metric on the cotangent bundle of Teichmüller space. The norm of a quadratic differential  $q$  is the  $L^2$ -norm of  $q$  with respect to the underlying hyperbolic metric on the base point of  $q$ . The unit cotangent bundle  $\tilde{\mathcal{Q}}_{WP}(S)$  for the metric projects to an orbifold bundle  $\mathcal{Q}_{WP}(S)$  over the moduli space of curves. The geodesic flow  $\Phi_{WP}^t$  for the metric acts on  $\mathcal{Q}_{WP}(S)$ , however this flow is incomplete. Periodic orbits for  $\Phi_{WP}^t$  are in bijection with conjugacy classes of pseudo-Anosov elements [DW03] and hence they are in bijection with periodic orbits for the Teichmüller flow. Each of these orbits supports an invariant Borel probability measure, and the set of all these measures is dense in the space  $\mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$  of  $\Phi_{WP}^t$ -invariant Borel probability measures on  $\mathcal{Q}_{WP}(S)$  equipped with the weak\*-topology [H!0b]. In other words, there is a natural bijection between dense subsets of  $\mathcal{M}_T(\mathcal{Q}(S))$  and  $\mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$ . The space  $\mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$  contains the *Lebesgue Liouville measure* for the Weil-Petersson metric. This measure was shown to be ergodic in [BMW12].

For a measure  $\mu \in \mathcal{M}_T(\mathcal{Q}(S))$  (or  $\nu \in \mathcal{M}_{WP}(\mathcal{Q}(S))$ ) let  $h(\mu)$  be the *entropy* of  $\mu$  (or of  $\nu$ ). The entropy of a measure  $\mu \in \mathcal{M}_T(\mathcal{Q}(S))$  is at most  $6g - 6 + 2m$  [H11]. In contrast, Brock, Masur and Minsky constructed invariant Borel probability measures for the Weil-Petersson geodesic flow whose entropy is arbitrarily large [BMM11]. The paper [PWW10] also contains some information on the dynamics of  $\Phi_{WP}^t$ .

**Definition.** A *measurable (or continuous) conjugacy* of a continuous flow  $\Psi^t$  on a topological space  $X$  into a continuous flow  $\Xi^t$  on a topological space  $Y$  is an injective measurable (or continuous) map  $\Lambda : X \rightarrow Y$  such that there is a measurable (or continuous) function  $\rho : X \times \mathbb{R} \rightarrow \mathbb{R}$  with the following properties.

- (1)  $\rho(x, 0) = 0$  for all  $x \in X$ .
- (2) For each fixed  $x \in X$  the function  $\rho(x, \cdot) : s \rightarrow \rho(x, s)$  is an increasing homeomorphism.
- (3)  $\Lambda(\Psi^t x) = \Xi^{\rho(x, t)} \Lambda(x)$  for all  $x \in X, t \in \mathbb{R}$ .

We show

**Theorem 1.** *There is a  $\Phi_T^t$ -invariant Borel subset  $\mathcal{E} \subset \mathcal{Q}(S)$  with the following properties.*

- (1)  $\mu(\mathcal{E}) = 1$  for every  $\Phi_T^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$ .
- (2) *There is a measurable conjugacy  $\Lambda : \mathcal{E} \rightarrow (\mathcal{Q}_{WP}(S), \Phi_{WP}^t)$  whose restriction to any compact invariant set is continuous.*
- (3)  $\Lambda$  induces a continuous injection

$$\Theta : \mathcal{M}_T(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{WP}(\mathcal{Q}_{WP}(S)).$$

- (4)  $\Theta$  is not surjective, but its image contains the Lebesgue Liouville measure.  
 (5)

$$\infty > h(\Theta(\mu)) \geq h(\mu)/\sqrt{2\pi(2g-2+n)} \text{ for all } \mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)).$$

To show that the map  $\Theta$  is not surjective we construct an explicit  $\Phi_{WP}^t$ -invariant Borel probability measure on  $\mathcal{Q}_{WP}(S)$  which is not contained in its image. However, this measure is not ergodic.

We also obtain information on  $\Phi_{WP}^t$ -invariant Borel probability measures which are not contained in the image of  $\Theta$ . For simplicity we restrict our attention to ergodic measures.

**Theorem 2.** *Let  $\nu$  be any  $\Phi_{WP}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}_{WP}(S)$ . Then there is an invariant Borel set  $A \subset \mathcal{Q}_{WP}(S)$  with  $\nu(A) = 1$ , and there is a measurable conjugacy*

$$\Xi : A \rightarrow (\mathcal{Q}_{\mathcal{T}}(S), \Phi_{\mathcal{T}}^t).$$

*The measure  $\Xi_*\nu$  determines a locally finite  $\Phi_{\mathcal{T}}^t$ -invariant Borel measure on  $\mathcal{Q}(S)$  which is finite if and only if  $\nu \in \Theta(\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)))$ .*

The strategy for the proof of Theorem 1 consists in a geometric comparison between biinfinite Teichmüller geodesics and biinfinite Weil-Petersson geodesics provided that these geodesics satisfy suitable recurrence properties under the action of the mapping class group. Particular such geodesics are geodesics which are contained in the thick part of Teichmüller space.

Denote for  $\epsilon > 0$  by  $\mathcal{T}(S)_{\epsilon}$  the  $\text{Mod}(S)$ -invariant subset of all hyperbolic metrics whose *systole* is at least  $\epsilon$ . The mapping class group acts cocompactly on  $\mathcal{T}(S)_{\epsilon}$  and hence by invariance, the restrictions to  $\mathcal{T}(S)_{\epsilon}$  of the Teichmüller metric and the Weil-Petersson metric are locally uniformly bilipschitz equivalent. It follows easily from the work of Masur and Minsky (see [R07b]) and Brock [B03] that on the large scale, the restrictions to  $\mathcal{T}(S)_{\epsilon}$  of the distances  $d_{\mathcal{T}}$  and  $d_{WP}$  induced by the Teichmüller metric and the Weil-Petersson metric, respectively, are not bilipschitz equivalent. For example, there is a sequence of points  $\{x_i\} \subset \mathcal{T}(S)_{\epsilon}$  and a number  $c > 0$  such that  $d_{\mathcal{T}}(x_0, x_i) \rightarrow \infty$  and  $d_{WP}(x_0, x_i) \leq c$ . The Teichmüller geodesics connecting  $x_0$  to  $x_i$  enter arbitrarily deeply into the thin part of Teichmüller space.

In [BMM11], Brock, Masur and Minsky showed that for every  $\epsilon > 0$ , biinfinite Teichmüller geodesics which are entirely contained in  $\mathcal{T}(S)_{\epsilon}$  are fellow-traveled by biinfinite Weil-Petersson geodesics, and biinfinite Weil-Petersson geodesics entirely contained in  $\mathcal{T}(S)_{\epsilon}$  are fellow-traveled by biinfinite Teichmüller geodesics.

The arguments used for the proof of Theorem 1 yield a similar result. For a formulation, we use the distance  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$  induced by the Teichmüller metric to define the *Hausdorff-distance*  $d_H(A, B) \in [0, \infty]$  between two subsets  $A, B$  of  $\mathcal{T}(S)$  as the infimum of all numbers  $r > 0$  such that  $A$  is contained in the  $r$ -neighborhood of  $B$  and  $B$  is contained in the  $r$ -neighborhood of  $A$ .

**Theorem 3.** *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property.*

- (1) *Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Weil-Petersson geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*
- (2) *Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\xi : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Weil-Petersson geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Teichmüller geodesic  $\gamma : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\xi(J), \gamma(J')) \leq R$ .*

The organization of this work is as follows. In Section 2 we establish the second part of Theorem 3 from standard properties of Teichmüller geodesics and some results of Brock, Masur and Minsky [BMM10].

Section 3 contains some geometric results on the Weil-Petersson metric. We begin with the fairly easy observation that given  $\epsilon > 0$  and a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$ , there are infinite Weil-Petersson geodesic rays  $\gamma_+, \gamma_- : [0, \infty) \rightarrow \mathcal{T}(S)$  which are uniform limits on compact sets of Weil-Petersson geodesic segments connecting  $\gamma(0)$  to points  $\gamma(T_i), \gamma(-R_i)$  for suitably chosen sequences  $T_i \rightarrow \infty, R_i \rightarrow \infty$ . Since the completion of Teichmüller space with respect to the Weil-Petersson metric is a CAT(0) geodesic metric space, Theorem 3 now predicts the existence of a biinfinite Weil-Petersson geodesic which is forward *asymptotic* to  $\gamma_+$  and backward asymptotic to  $\gamma_-$ . However, the curvature of the Weil-Petersson metric is not bounded from above by a negative constant, and the existence of such a geodesic is not immediate.

With an argument based on ruled surfaces and angle comparison we derive a sufficient condition for the existence of a biinfinite Weil-Petersson geodesic which is forward and backward asymptotic to given Weil-Petersson geodesic rays. Roughly speaking, this condition is satisfied if the geodesic rays spend a sufficient (but finite) amount of time in the thick part of Teichmüller space.

In Section 4 we find a sufficient condition for a Weil-Petersson geodesic segment of uniformly bounded length to pass through the thick part of Teichmüller space. This condition is a consequence of a quantitative version of the following result of Wolpert [W03]: If  $\zeta : [0, R] \rightarrow \mathcal{T}(S)$  is any Weil-Petersson geodesic of uniformly bounded length and if there is a simple closed curve  $\alpha$  which is long at both endpoints of  $\zeta$  and becomes very short along  $\zeta$ , then  $\zeta$  twists about  $\alpha$  or about a curve  $\beta$  which is disjoint from  $\alpha$ .

Section 5 contains the main technical results of this work. We use hyperbolicity of the *curve graph* and some of the combinatorial tools introduced by Masur and Minsky [MM00] to establish a sufficient condition for a Weil-Petersson geodesic segment of arbitrary length to spend a fixed proportion of time in the thick part of Teichmüller space. This result is used in Section 6 to construct the Borel set  $\mathcal{E}$  and the conjugacy  $\Lambda : \mathcal{E} \rightarrow \mathcal{Q}_{WP}(S)$  from Theorem 1. The first part of Theorem 3 follows easily.

In Section 7 we show that the conjugacy  $\Lambda$  induces an injection  $\Theta$  of the space of invariant Borel probability measures on  $\mathcal{Q}(S)$  into the space of invariant Borel probability measures on  $\mathcal{Q}_{WP}(S)$ . In Section 8 we prove Theorem 2 and use this to characterize the image of the map  $\Theta$ . We also construct an example of a measure

which is not contained in the image of the map  $\Theta$ . Finally in Section 9 we find that the image of  $\Theta$  contains the Lebesgue Liouville measure.

**Notation:** Throughout the paper, we write  $f \asymp g$  for two nonvanishing functions  $f, g$  if  $f/g$  and  $g/f$  are uniformly bounded.

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## 2. WEIL-PETERSSON GEODESICS IN THE THICK PART OF $\mathcal{T}(S)$

Let  $S$  be an oriented surface of genus  $g \geq 0$  with  $m \geq 0$  punctures and  $3g-3+m \geq 2$  and let  $\mathcal{T}(S)$  be the Teichmüller space for  $S$ . In this section we investigate geodesics for the Weil-Petersson metric on  $\mathcal{T}(S)$  which remain entirely in a fixed thick part of Teichmüller space. Our goal is to prove the second part of Theorem 3 which relates such a geodesic to a Teichmüller geodesic.

We begin with summarizing those properties of the Weil-Petersson metric which are needed in the sequel. More information and references are contained in the survey paper [W03].

A geodesic for the Weil-Petersson metric is also called a WP-geodesic. Such a geodesic will always be parametrized by arc length. A *WP-ray* in  $\mathcal{T}(S)$  is a WP-geodesic  $\gamma : [0, T) \rightarrow \mathcal{T}(S)$  for some  $T \in (0, \infty]$  which can not be extended, i.e. which leaves every compact set.

A *geodesic lamination* for a complete hyperbolic structure on  $S$  of finite volume is a *compact* subset of  $S$  which is foliated into simple geodesics. A geodesic lamination  $\lambda$  on  $S$  is called *minimal* if each of its half-leaves is dense in  $\lambda$ . Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. A geodesic lamination  $\lambda$  is said to *fill*  $S$  if every simple closed geodesic on  $S$  intersects  $\lambda$  transversely. This is equivalent to stating that the complementary components of  $\lambda$  are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination  $\lambda$  together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in  $S$  which intersects  $\lambda$  nontrivially and transversely and whose endpoints are contained in complementary regions of  $\lambda$ . The geodesic lamination  $\lambda$  is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Every minimal geodesic lamination is the support of a measured geodesic lamination. The space  $\mathcal{ML}$  of measured geodesic laminations on  $S$  can be equipped with the weak\*-topology. Its projectivization  $\mathcal{PML}$  is called

the space of *projective measured geodesic laminations*, and it is homeomorphic to the sphere  $S^{6g-7+2m}$ .

For every marked hyperbolic metric  $h \in \mathcal{T}(S)$ , every essential free homotopy class  $\alpha$  on  $S$  can be represented by a closed  $h$ -geodesic which is unique up to parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The  $h$ -length  $\ell_\alpha(h)$  of the class  $\alpha$  is the length of its geodesic representative. Equivalently,  $\ell_\alpha(h)$  equals the minimum of the  $h$ -lengths of all closed curves representing the class  $\alpha$ .

Length of simple closed curves extends to a continuous *length function*  $\mathcal{T}(S) \times \mathcal{ML} \rightarrow (0, \infty)$  which assigns to a hyperbolic metric  $h \in \mathcal{T}(S)$  and a measured geodesic lamination  $\mu$  the  $h$ -length  $\ell_\mu(h)$  of  $\mu$ . This length function satisfies  $\ell_{a\mu}(h) = a\ell_\mu(h)$  for all  $h \in \mathcal{T}(S)$ ,  $\mu \in \mathcal{ML}$  and every  $a > 0$ . For every fixed  $h \in \mathcal{T}(S)$ , the set of all measured geodesic laminations of  $h$ -length one is a section of the projection  $\mathcal{ML} \rightarrow \mathcal{PML}$ .

The following result is due to Wolpert (Corollary 4.7 of [W87]) and is of fundamental importance for this work.

**Theorem 2.1.** *For any  $\mu \in \mathcal{ML}$  and for every Weil-Petersson geodesic  $\gamma : [a, b] \rightarrow \mathcal{T}(S)$  the function  $t \rightarrow \ell_\mu(\gamma(t))$  is convex.*

There is a continuous pairing  $i : \mathcal{ML} \times \mathcal{ML} \rightarrow [0, \infty)$ , the so-called *intersection form*, which extends the geometric intersection number between simple closed curves (see [Bo86] for this result of Thurston). Two measured geodesic laminations  $\mu, \nu$  bind  $S$  if  $i(\zeta, \mu) + i(\zeta, \nu) > 0$  for every  $\zeta \in \mathcal{ML}$ . Pairs  $(\mu, \nu)$  of measured geodesic laminations which bind  $S$  and which satisfy  $i(\mu, \nu) = 1$  correspond precisely to area one quadratic differentials for  $S$  via a map which associates to an area one quadratic differential  $q$  the ordered pair  $(q^v, q^h)$  composed of its vertical and its horizontal measured geodesic lamination, respectively.

A *pants decomposition* for  $S$  is a collection of  $3g - 3 + m$  pairwise disjoint simple closed essential curves on  $S$  which decompose  $S$  into  $2g - 2 + m$  *pairs of pants*. Here by a pair of pants we mean a surface which is homeomorphic to a three-holed sphere. By a classical result of Bers (see [B92]), there is a number  $\chi_0 > 0$  only depending on the topological type of  $S$  such that for every complete hyperbolic metric  $h$  on  $S$  of finite volume, there is a pants decomposition for  $S$  consisting of simple closed curves of  $h$ -length at most  $\chi_0$ . A number  $\chi_0 > 0$  with this property is called a *Bers constant* for  $S$ , and a simple closed curve of  $h$ -length at most  $\chi_0$  is called a *Bers curve* for  $h$ . Such a curve supports a unique projective measured lamination. A pants decomposition which consists of Bers curves for  $h$  will be called a *Bers decomposition* for  $h$ .

We begin with recalling from [BMM10] (Definition 2.5) the definition of an ending measure for a WP-ray.

**Definition 2.2.** A *projective ending measure*  $[\mu] \in \mathcal{PML}$  for a WP-geodesic ray  $\gamma : [0, T) \rightarrow \mathcal{T}(S)$  ( $T \in (0, \infty]$ ) is any limit in  $\mathcal{PML}$  of the projective classes of any infinite sequence of distinct Bers curves for  $\gamma$ .

The following theorem combines Corollary 2.12 and Proposition 4.4 of [BMM10]. For its formulation, for a number  $\epsilon > 0$  denote by  $\mathcal{T}(S)_\epsilon$  the set of all points whose systole is at least  $\epsilon$ . Call an infinite WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  *recurrent* if there is some  $\epsilon > 0$  and an unbounded sequence  $(t_i) \subset [0, \infty)$  such that  $\gamma(t_i) \in \mathcal{T}(S)_\epsilon$ . Here as in the introduction,  $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$  is the subset of all surfaces whose systole is at least  $\epsilon$ .

**Theorem 2.3.** *Let  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  be a recurrent WP-geodesic ray.*

- (1) *Any two ending measures for  $\gamma$  have the same support.*
- (2) *The support of an ending measure for  $\gamma$  is a minimal geodesic lamination which fills  $S$ .*
- (3) *If  $\mu \in \mathcal{ML}$  is any measured geodesic lamination whose length along  $\gamma$  is bounded then the support of  $\mu$  equals the support of an ending measure for  $\gamma$ .*

Unlike in the case of Teichmüller geodesics, however, there are recurrent WP-rays so that the support of an ending measure is not *uniquely ergodic*, i.e. it admits more than one transverse measure up to scale [BMo14].

The main tool for the proof of the second part of Theorem 3 from the introduction is the *curve graph*  $\mathcal{CG}(S)$  of  $S$ . The vertex set of this graph is the set  $\mathcal{C}(S)$  of all free homotopy classes of unoriented *essential* simple closed curves on  $S$ , i.e. simple closed curves which are neither contractible nor freely homotopic into a puncture. Two vertices are joined by an edge if and only if the corresponding free homotopy classes can be realized disjointly. Since  $3g - 3 + m \geq 2$  by assumption,  $\mathcal{CG}(S)$  is connected (see [MM99] and the references given there). In the sequel we often do not distinguish between an essential simple closed curve  $\alpha$  on  $S$  and the vertex of the curve graph defined by  $\alpha$ .

Providing each edge in  $\mathcal{CG}(S)$  with the standard euclidean metric of diameter 1 equips the curve graph with a geodesic metric  $d_{\mathcal{C}}$ . However,  $\mathcal{CG}(S)$  is not locally finite and therefore the metric space  $(\mathcal{CG}(S), d_{\mathcal{C}})$  is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely,  $\mathcal{CG}(S)$  is hyperbolic of infinite diameter. The mapping class group naturally acts on  $\mathcal{CG}(S)$  as a group of simplicial isometries.

Define a map

$$\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$$

by associating to a complete hyperbolic metric  $h$  on  $S$  of finite volume a Bers curve  $\Upsilon_{\mathcal{T}}(h) \in \mathcal{C}(S)$ . Note that such a map is not unique. The ambiguity in this definition is uniformly controlled [MM99] (or see Lemma 2.1 of [H10a]).

**Lemma 2.4.** *For every  $\chi > 0$  there is a number  $a(\chi) > 0$  with the following property. Let  $h \in \mathcal{T}(S)$  and let  $\alpha, \beta$  be two simple closed curves of  $h$ -length at most  $\chi$ . Then  $d_{\mathcal{C}}(\alpha, \beta) \leq a(\chi)$ .*

Masur and Minsky [MM99] showed that  $\Upsilon_{\mathcal{T}}$  is coarsely Lipschitz with respect to the Teichmüller distance  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$  and the distance  $d_{\mathcal{C}}$  on the curve graph. We use the version which is explicitly stated as Lemma 2.2 in [H10a].



**Lemma 2.5.** *There is a number  $L_0 > 1$  such that*

- (1)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L_0 d_{\mathcal{T}}(g, h) + L_0$  for all  $g, h \in \mathcal{T}(S)$ .
- (2)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\varphi h), \varphi \Upsilon_{\mathcal{T}}(h)) \leq L_0$  for all  $h \in \mathcal{T}(S), \varphi \in \text{Mod}(S)$ .

Let  $J \subset \mathbb{R}$  be a closed connected set. For a number  $L > 1$ , a map  $\gamma : J \rightarrow \mathcal{CG}(S)$  is an  $L$ -quasi-geodesic if

$$|t - s|/L - L \leq d_{\mathcal{C}}(\gamma(s), \gamma(t)) \leq L|t - s| + L \text{ for all } s, t \in J.$$

A map  $\gamma : J \rightarrow \mathcal{CG}(S)$  is an *unparametrized  $L$ -quasi-geodesic* if there is a closed connected set  $I \subset \mathbb{R}$  and a homeomorphism  $\rho : I \rightarrow J$  such that  $\gamma \circ \rho : I \rightarrow \mathcal{CG}(S)$  is an  $L$ -quasi-geodesic.

The following result of Masur and Minsky (Theorem 2.3 and Theorem 2.6 of [MM99]) is essential for the proof of Theorem 1.

**Theorem 2.6.** *There is a number  $L_1 > 1$  such that the image under  $\Upsilon_{\mathcal{T}}$  of every Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  is an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ .*

In the case that the Teichmüller geodesic  $\gamma$  remains entirely in the  $\epsilon$ -thick part of Teichmüller space, the path  $\Upsilon_{\mathcal{T}}(\gamma)$  is in fact a *parametrized  $L'$ -quasi-geodesic* where  $L' > 0$  only depends on  $\epsilon$  [H10a].

We do not know whether the image under the map  $\Upsilon_{\mathcal{T}}$  of a Weil-Petersson geodesic is an unparametrized quasi-geodesic in  $\mathcal{CG}(S)$ . But we can use ending measures to show that the image under  $\Upsilon_{\mathcal{T}}$  of a Weil-Petersson geodesic ray which is entirely contained in  $\mathcal{T}(S)_{\epsilon}$  for some  $\epsilon > 0$  makes a definite progress in  $\mathcal{CG}(S)$  in uniformly bounded time.

**Lemma 2.7.** *For every  $\epsilon > 0$  and every  $R > 0$  there is a number  $T_0 = T_0(\epsilon, R) > 0$  with the following property. Let  $b \geq T_0$  and let  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_{\epsilon}$  be a Weil-Petersson geodesic. Then  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(b)) \geq R$ .*

*Proof.* Assume to the contrary that there is some  $\epsilon > 0$  and some  $R > 0$  such that there is no  $T_0(\epsilon, R) > 0$  with the properties stated in the lemma. Then there is for every  $n > 0$  a number  $T_n > n$  and a WP-geodesic  $\gamma_n : [0, T_n] \rightarrow \mathcal{T}(S)_{\epsilon}$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma_n(0)), \Upsilon_{\mathcal{T}}(\gamma_n(T_n))) \leq R$ .

By coarse equivariance of the map  $\Upsilon_{\mathcal{T}}$  under the action of the mapping class group (part (2) of Lemma 2.5) and cocompactness of the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_{\epsilon}$ , we may assume that the WP-geodesics  $\gamma_n$  issue from the same compact set  $K \subset \mathcal{T}(S)_{\epsilon}$ . Since  $\gamma_n \subset \mathcal{T}(S)_{\epsilon}$  for all  $n$ , after passing to a subsequence we may assume that the geodesics  $\gamma_n$  converge uniformly on compact sets to a WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)_{\epsilon}$ .

For  $n > 0$  let  $\alpha_n \in \mathcal{ML}$  be the measured geodesic lamination with  $\ell_{\alpha_n}(\gamma_n(0)) = 1$  which is contained in the projective class of the curve  $\Upsilon_{\mathcal{T}}(\gamma_n(T_n))$ , viewed as a projective measured geodesic lamination. Since  $\gamma_n(0) \in K \subset \mathcal{T}(S)_{\epsilon}$ , the systole of the metric  $\gamma_n(0)$  is at least  $\epsilon$ . Therefore the lamination  $\alpha_n$  is obtained by multiplying

the simple closed curve  $\Upsilon_{\mathcal{T}}(\gamma_n(T_n))$  with a weight which is bounded from above by  $1/\epsilon$ . Now the  $\gamma_n(T_n)$ -length of the simple closed curve  $\Upsilon_{\mathcal{T}}(\gamma_n(T_n))$  is at most  $\chi_0$  and hence the  $\gamma_n(T_n)$ -length of  $\alpha_n$  does not exceed  $\chi_0/\epsilon$ . By convexity of the length function along WP-geodesics, this implies that the length of  $\alpha_n$  along  $\gamma_n[0, T_n]$  is uniformly bounded, independent of  $n$ .

The set

$$\{\alpha \in \mathcal{ML} \mid \ell_{\alpha}(x) = 1 \text{ for some } x \in K\}$$

is compact. Thus by passing to a subsequence we may assume that the measured geodesic laminations  $\alpha_n$  converge as  $n \rightarrow \infty$  to a measured geodesic lamination  $\alpha$ . By continuity of the length function, we have  $\ell_{\alpha}(\gamma(0)) = 1$ . Moreover, the length of  $\alpha$  along  $\gamma$  is uniformly bounded. Namely, by convexity, for each  $T > 0$  and each  $n$  which is sufficiently large that  $T_n > T$ , we have

$$\ell_{\alpha_n}(\gamma_n(T)) \leq \chi_0/\epsilon.$$

Since  $\alpha_n \rightarrow \alpha$  weakly and  $\gamma_n(T) \rightarrow \gamma(T)$  as  $n \rightarrow \infty$ , continuity of the length function yields  $\ell_{\alpha}(\gamma(T)) \leq \chi_0/\epsilon$  as well. Now  $T > 0$  was arbitrary and therefore the length of  $\alpha$  is uniformly bounded along  $\gamma$ .

We claim that the support of  $\alpha$  does not fill up  $S$ . This is equivalent to stating that there is a measured geodesic lamination  $\nu$  whose support does not coincide with the support of  $\alpha$  and such that  $i(\nu, \alpha) = 0$ .

To see that this is indeed the case, note that since  $\gamma_n(0) \rightarrow \gamma(0)$ , by coarse continuity of the map  $\Upsilon_{\mathcal{T}}$  (part (1) of Lemma 2.5) and by our assumption that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma_n(0)), \Upsilon_{\mathcal{T}}(\gamma_n(T_n))) \leq R \text{ for all } n,$$

the distance in  $\mathcal{CG}(S)$  between  $\Upsilon_{\mathcal{T}}(\gamma_n(T_n))$  and  $c_0 = \Upsilon_{\mathcal{T}}(\gamma(0))$  is bounded independent of  $n$ . By passing to a subsequence we may assume that this distance equals a fixed number  $k \geq 0$ . Then for each  $n$  in the subsequence, there is a collection  $c_n^0, \dots, c_n^k \subset \mathcal{ML}$  of weighted simple closed geodesics of (weighted) length one for the metric  $\gamma(0) \in \mathcal{T}(S)$  such that

$$i(c_n^j, c_n^{j+1}) = 0 \text{ for every } j < k$$

(here as before,  $i$  is the intersection form) and that  $[c_0] = [c_n^0]$  and  $[c_n^k] = [\alpha_n] = [\Upsilon_{\mathcal{T}}(\gamma_n(T_n))]$  for all  $n$  (where  $[\mu]$  denotes the projective class of  $\mu \in \mathcal{ML}$ ). By passing to another subsequence, we may assume that for each  $j$  the measured geodesic laminations  $c_n^j$  converge as  $n \rightarrow \infty$  to a measured geodesic lamination  $\nu_j$ . By continuity of the intersection form, we have  $i(\nu_j, \nu_{j+1}) = 0$  for all  $j$ . Since  $\nu_k = \alpha = \lim_{n \rightarrow \infty} \alpha_n$  and since  $[\nu_0] = [c_0]$ , if the support of  $\alpha$  fills  $S$  then the supports of the laminations  $\nu_j$  have to coincide with the support of  $\alpha$ . But  $[c_0^n] = [c_0]$  for all  $n$  and the support of  $c_0$  is a simple closed curve and hence this is impossible (compare [MM99] for this argument of Luo). As a consequence, the support of  $\alpha$  does not fill  $S$ .

However, the WP-ray  $\gamma$  is entirely contained in  $\mathcal{T}(S)_{\epsilon}$ , in particular it is recurrent. Since the length of  $\alpha$  is bounded along  $\gamma$ , this violates the second part of Theorem 2.3. The lemma follows from this contradiction.  $\square$

As in the introduction, let  $d_H$  be the Hausdorff distance for subsets of  $\mathcal{T}(S)$  with respect to the Teichmüller metric. We use Lemma 2.7 and an idea of Mosher [Mo03] to show the second part of Theorem 3 from the introduction. For its formulation, denote as in the introduction by  $\mathcal{Q}_{WP}(S)$  the moduli space of quadratic differentials of Weil-Petersson norm one. Recall moreover the definition of a continuous conjugacy of two flows on topological spaces.

**Proposition 2.8.** (1) *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property. Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Weil-Petersson geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Teichmüller geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*

(2) *Let  $C \subset \mathcal{Q}_{WP}(S)$  be a compact set which is invariant under the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric. Then there is a continuous conjugacy  $\Psi : C \rightarrow \mathcal{Q}(S)$  of the restriction of  $\Phi_{WP}^t$  to  $C$  into the Teichmüller geodesic flow.*

*Proof.* Let  $\epsilon > 0$ , let  $a(\chi_0) > 1$  be as in Lemma 2.4 and let  $T_0 = T_0(\epsilon, 2a(\chi_0)+3) > 0$  be as in Lemma 2.7. Note that  $T_0$  only depends on  $\epsilon$ .

Unit balls in the cotangent bundle of  $\mathcal{T}(S)$  for both the Teichmüller metric and the Weil-Petersson metric depend continuously on the basepoint. Thus by invariance under the action of the mapping class group and cocompactness of the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_\epsilon$ , the restriction to  $\mathcal{T}(S)_\epsilon$  of the Weil-Petersson metric is locally uniformly bilipschitz equivalent to the restriction of the Teichmüller metric. Hence there is a number  $L = L(\epsilon) > 1$  such that  $d_{\mathcal{T}}(\gamma(0), \gamma(b)) \leq Lb$  for any WP-geodesic  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_\epsilon$  where as before,  $d_{\mathcal{T}}$  is the distance induced by the Teichmüller metric. As a consequence, for every  $b \leq T_0$ , every WP-geodesic  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_\epsilon$  is entirely contained in the ball of radius  $LT_0$  about  $\gamma(0)$  for the Teichmüller metric and therefore  $d_H(\gamma[0, b], \gamma(0)) \leq LT_0$ . Thus it is enough to show the proposition for Weil-Petersson geodesics in  $\mathcal{T}(S)_\epsilon$  of length at least  $T_0$ .

By Lemma 2.7, if  $\gamma : [b, c] \rightarrow \mathcal{T}(S)_\epsilon$  is a WP-geodesic of length  $c - b \geq T_0$  then  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(b), \Upsilon_{\mathcal{T}}\gamma(c)) \geq 2a(\chi_0) + 3$  and hence by Lemma 2.4, the  $\gamma(b)$ -length of any simple closed curve  $\alpha \in \mathcal{C}(S)$  with  $\ell_\alpha(\gamma(c)) \leq \chi_0$  is bigger than  $\chi_0$ . In particular, by convexity of length functions along Weil-Petersson geodesics, we have  $\ell_\alpha(\gamma(t)) \leq \ell_\alpha(\gamma(b))$  for every  $t \in [b, c]$ .

For the WP-geodesic  $\gamma : [b, c] \rightarrow \mathcal{T}(S)_\epsilon$  we say that the projective measured geodesic lamination  $[\alpha]$  defined by a simple closed curve  $\alpha \in \mathcal{C}(S)$  is *realized* at the right endpoint  $c$  of the parameter interval  $[b, c]$  if the  $\gamma(c)$ -length of  $\alpha$  does not exceed  $\chi_0$ . By the above, we then have  $\ell_\alpha(\gamma(t)) \leq \ell_\alpha(\gamma(b))$  for every  $t \in [b, c]$ . If  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$  is an infinite WP-geodesic ray then the projectivization  $[\lambda] \in \mathcal{PM}\mathcal{L}$  of a measured geodesic lamination  $\lambda$  is *realized* at the right endpoint  $\infty$  if the length of  $\lambda$  along  $\gamma[0, \infty)$  assumes its maximum at  $\gamma(0)$ . By Theorem 2.3, Lemma 2.7 and the above discussion, an ending measure for  $\gamma$  is realized at the right endpoint of  $J = [0, \infty)$ , moreover any projective measured geodesic lamination which is realized at  $\infty$  is supported in the support of an ending measure.

Using an idea of Mosher [Mo03], define  $\Gamma_\epsilon$  to be the set of all triples  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \mu_+, \mu_-)$  with the following properties.

- (1)  $J \subset \mathbb{R}$  is a closed connected set of diameter at least  $T_0$  containing 0.
- (2)  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is a Weil-Petersson geodesic.
- (3)  $\mu_+, \mu_- \in \mathcal{ML}$  are measured geodesic laminations of  $\gamma(0)$ -length one whose projectivizations  $[\mu_+], [\mu_-]$  are realized at the right and left endpoint of  $J$ , respectively.

We equip  $\Gamma_\epsilon$  with the product topology, using the weak\*-topology on  $\mathcal{ML}$  for the second and third factor and the compact open topology for the arcs  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . The mapping class group  $\text{Mod}(S)$  naturally acts diagonally on  $\Gamma_\epsilon$ .

We follow Mosher (Proposition 3.17 of [Mo03]) and show that the action of  $\text{Mod}(S)$  on  $\Gamma_\epsilon$  is cocompact. Since  $\text{Mod}(S)$  acts isometrically and cocompactly on  $\mathcal{T}(S)_\epsilon$ , for this it is enough to show that the subset of  $\Gamma_\epsilon$  consisting of all triples with the additional property that  $\gamma(0)$  is contained in a fixed compact subset  $A$  of  $\mathcal{T}(S)_\epsilon$  is compact. Now the topology of  $\Gamma_\epsilon$  is metrizable and hence this follows if every sequence in  $\Gamma_\epsilon$  contained in the subset  $\{(\gamma : J \rightarrow \mathcal{T}(S), \mu_+, \mu_-) \in \Gamma_\epsilon \mid \gamma(0) \in A\}$  has a convergent subsequence.

By the Arzela Ascoli theorem (or simply by properties of geodesics for the Weil-Petersson metric), the set of geodesic arcs  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  where  $J \subset \mathbb{R}$  is a closed connected subset containing 0 and such that  $\gamma(0) \in A$  is compact with respect to the compact open topology. As the length function is continuous on  $\mathcal{T}(S) \times \mathcal{ML}$ , it is enough to show that the following holds. Let  $\gamma_i : J_i \rightarrow \mathcal{T}(S)_\epsilon$  ( $i > 0$ ) be a sequence of Weil-Petersson geodesics which converge locally uniformly to  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . For each  $i$  let  $\mu_i$  be a measured geodesic lamination of  $\gamma_i(0)$ -length one whose projectivization  $[\mu_i]$  is realized at the right endpoint of  $J_i$ . If  $\mu_i \rightarrow \mu \in \mathcal{ML}$ , then the projectivization  $[\mu]$  of  $\mu$  is realized at the right endpoint of  $J$ .

Assume first that  $J \cap [0, \infty) = [0, b]$  for some  $b \in (0, \infty)$ . Then for sufficiently large  $i$  we have  $J_i \cap [0, \infty) = [0, b_i]$  with  $b_i \in (0, \infty)$  and  $b_i \rightarrow b$ . Thus  $\gamma_i(b_i) \rightarrow \gamma(b)$  ( $i \rightarrow \infty$ ) and therefore by continuity of length functions and the collar lemma, there is only a *finite* number of simple closed curves  $\alpha \in \mathcal{C}(S)$  of length at most  $\chi_0$  with respect to one of the metrics  $\gamma_i(b_i), \gamma(b)$ . Hence by passing to a subsequence we may assume that there is a curve  $\alpha \in \mathcal{C}(S)$  such that  $[\mu_i] = [\alpha] = [\mu]$  for all sufficiently large  $i$ . The  $\gamma_i(b_i)$ -length of  $\alpha$  is at most  $\chi_0$  for all sufficiently large  $i$  and hence by continuity of length functions, the same is true for the  $\gamma(b)$ -length of  $\alpha$ . In other words, the limit  $[\mu] \in \mathcal{PML}$  of the sequence  $[\mu_i]$  is realized at the right endpoint  $b$  of  $J$ .

The same argument is also valid if the right endpoint of  $J$  is infinite. Namely, assume first that  $J_i \cap [0, \infty) = [0, b_i]$  for some  $b_i > 0$  with  $b_i \rightarrow \infty$  ( $i \rightarrow \infty$ ). By the above discussion, for sufficiently large  $i$  (namely, for all  $i$  such that  $b_i > T_0$ ) the length of  $\mu_i$  along  $\gamma_i[0, b_i]$  assumes its maximum at  $\gamma_i(0)$ . Thus if  $T > 0$  is arbitrary and if  $i > 0$  is sufficiently large that  $b_i > \max\{T_0, T\}$  then  $\ell_{\mu_i}(\gamma_i(T)) \leq \ell_{\mu_i}(\gamma_i(0))$ . Since  $\gamma_i(0) \rightarrow \gamma(0)$  and  $\gamma_i(T) \rightarrow \gamma(T)$  and  $\mu_i \rightarrow \mu$ , continuity of the length function implies that  $\ell_\mu(\gamma(T)) \leq \ell_\mu(\gamma(0))$ . Now  $T > 0$  was arbitrary and therefore the length of  $\mu$  along  $\gamma$  assumes its maximum at  $\gamma(0)$ . However, this just means that

the projectivization  $[\mu]$  is realized at the right infinite endpoint of  $J$ . The case that  $b_i = \infty$  for infinitely many  $i$  follows in the same way.

Any two simple closed curves  $\alpha, \beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha, \beta) \geq 3$  bind  $S$ . Thus by Theorem 2.3, Lemma 2.7 and the choice of  $T_0$ , for any  $(\gamma : J \rightarrow \mathcal{T}(S), \mu_+, \mu_-) \in \Gamma_{\epsilon}$  the measured geodesic laminations  $\mu_+, \mu_-$  bind  $S$ . These laminations then determine up to parametrization a Teichmüller geodesic  $\eta([\mu_+], [\mu_-])$  whose vertical and horizontal projective measured geodesic laminations are just the classes  $[\mu_+], [\mu_-]$ .

Let  $\sigma(\gamma, \mu_+, \mu_-)$  be the unique point on  $\eta([\mu_+], [\mu_-])$  which is the foot-point of the quadratic differential with vertical and horizontal measured geodesic laminations  $\mu_+/\sqrt{i(\mu_+, \mu_-)}, \mu_-/\sqrt{i(\mu_+, \mu_-)}$ . By continuity of the length function and the intersection form, the map taking  $(\gamma : J \rightarrow \mathcal{T}(S)_{\epsilon}, \mu_+, \mu_-) \in \Gamma_{\epsilon}$  to  $(\gamma(0), \sigma(\gamma, \mu_+, \mu_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$  is continuous. Moreover by construction, this map is equivariant with respect to the natural diagonal action of  $\text{Mod}(S)$  on  $\Gamma_{\epsilon}$  and on  $\mathcal{T}(S) \times \mathcal{T}(S)$ . Since the action of  $\text{Mod}(S)$  on  $\Gamma_{\epsilon}$  is cocompact, the same is true for the action of  $\text{Mod}(S)$  on the image of this map. Thus the Teichmüller distance between  $\gamma(0)$  and  $\sigma(\gamma, \mu_+, \mu_-)$  is bounded from above by a universal constant  $b > 0$ .

Let again  $(\gamma : J \rightarrow \mathcal{T}(S)_{\epsilon}, \mu_+, \mu_-) \in \Gamma_{\epsilon}$ . For each  $s \in J$  define

$$a_-(s) = \frac{1}{\ell_{\mu_-}(\gamma(s))}, \quad a_+(s) = \frac{1}{\ell_{\mu_+}(\gamma(s))}.$$

Let moreover  $\gamma^s(t) = \gamma(t + s)$ . Then the ordered triple  $(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-)$  lies in the  $\text{Mod}(S)$ -cocompact set  $\Gamma_{\epsilon}$  and hence the distance between  $\gamma(s)$  and the point  $\sigma(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-) \in \eta([\mu_+], [\mu_-])$  is at most  $b$ . As a consequence,  $\gamma(J)$  is contained in the  $b$ -neighborhood of the geodesic  $\eta([\mu_+], [\mu_-])$ .

Now  $s \in J$  was arbitrary and the ordered triple  $(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-)$  depends continuously on  $s$  with respect to the topology of  $\Gamma_{\epsilon}$ . Hence  $\gamma(J)$  is contained in the  $b$ -neighborhood of a suitably chosen subarc of  $\eta([\mu_+], [\mu_-])$ . Moreover, the map  $\gamma(s) \rightarrow \sigma(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-)$  is continuous in  $s$ . This means that the image subarc is contained in the  $b$ -neighborhood of  $\gamma(J)$ . Together we showed that the Hausdorff distance between  $\gamma(J)$  and a subarc of  $\eta([\mu_+], [\mu_-])$  is at most  $b$ . The first part of the proposition is proven.

The results obtained so far show that if  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_{\epsilon}$  is any biinfinite Weil-Petersson geodesic and if  $[\mu_+], [\mu_-]$  are ending measures for  $\gamma_+ = \gamma[0, \infty), \gamma_- = \gamma(-\infty, 0]$  then the Teichmüller geodesic  $\eta([\mu_+], [\mu_-])$  defined by a quadratic differential with vertical measured geodesic lamination in the class  $[\mu_+]$  and horizontal measured geodesic lamination in the class  $[\mu_-]$  is a uniform fellow-traveler of  $\gamma$ , measured with respect to the Teichmüller metric. In particular, there is a number  $\kappa > 0$  only depending on  $\epsilon$  such that  $\eta([\mu_+], [\mu_-])$  is contained in  $\mathcal{T}(S)_{\kappa}$ . Hence by a result of Masur [M82], the projective measured geodesic laminations  $[\mu_+], [\mu_-]$  are uniquely ergodic. Theorem 2.3 then implies that a projective ending measure for a subray of  $\gamma$  is unique.

Let  $\mu_+(\gamma), \mu_-(\gamma) \in \mathcal{ML}$  be the representative of the forward and backward projective ending measures whose  $\gamma(0)$ -length equals one. We claim that  $\mu_+(\gamma)$  and  $\mu_-(\gamma)$  depend continuously on  $\gamma$ . Namely, assume that  $\gamma_i \rightarrow \gamma$  uniformly on

compact sets and that  $\beta$  is a weak limit of the sequence  $(\mu_+(\gamma_i))$ . Then by continuity of length functions we have  $\ell_\beta(\gamma(0)) = 1$ , moreover we conclude as above (see also the proof of Lemma 2.7) that the length of  $\beta$  is bounded along  $\gamma[0, \infty)$ . Since these two properties determine the measured geodesic lamination  $\mu_+(\gamma)$  uniquely, continuous dependence of  $\mu_+(\gamma)$  on  $\gamma$  is immediate. Continuous dependence of  $\mu_-(\gamma)$  on  $\gamma$  follows in the same way.

Let  $C \subset \mathcal{Q}_{WP}(S)$  be a compact set which is invariant under the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric. Let  $\tilde{C}$  be the preimage of  $C$  in the space of quadratic differentials of Weil-Petersson norm one. By compactness of  $C$  there is a number  $\epsilon > 0$  such that for every  $q \in \tilde{C}$  the Weil-Petersson geodesic  $\gamma_q$  with initial velocity  $q$  is entirely contained in  $\mathcal{T}(S)_\epsilon$ .

For  $q \in \tilde{C}$  define  $\tilde{\Psi}(q) \in \tilde{\mathcal{Q}}(S)$  to be the unique area one quadratic differential with vertical and horizontal measured geodesic lamination

$$\mu_+(\gamma_q)/\sqrt{i(\mu_+(\gamma_q), \mu_-(\gamma_q))}, \quad \mu_-(\gamma_q)/\sqrt{i(\mu_+(\gamma_q), \mu_-(\gamma_q))},$$

respectively. Here  $\mu_+(\gamma_q)$  (or  $\mu_-(\gamma_q)$ ) is as before the forward (or backward) ending measure for the geodesic  $\gamma_q$  with initial velocity  $q$ . Then  $q \rightarrow \tilde{\Psi}(q)$  is continuous and equivariant with respect to the action of the mapping class group and hence it projects to a map  $\Psi : C \rightarrow \mathcal{Q}(S)$ . By convexity of length functions along WP-geodesics, the length of  $\mu_+(\gamma_q)$  is strictly decreasing along  $\gamma_q$ , and the length of  $\mu_-(\gamma_q)$  is strictly increasing. This implies that the restriction of  $\tilde{\Psi}$  to the unit cotangent line  $\gamma'_q$  of the WP-geodesic  $\gamma_q$  is a homeomorphism onto the unit cotangent line of the Teichmüller geodesic  $\tilde{\gamma}$  with initial velocity  $\tilde{\gamma}'(0) = \tilde{\Psi}(q)$ . Therefore the map  $\Psi$  defines a continuous conjugacy of the Weil-Petersson flow on  $C$  into the Teichmüller flow as defined in the introduction. This shows the second part of the proposition.  $\square$

### 3. ASYMPTOTIC RAYS FOR THE WEIL-PETERSSON METRIC

The goal of this section is to establish some differential geometric properties of Weil-Petersson geodesic rays which are needed for the proof of Theorem 1 from the introduction.

Using the assumptions and notations from Section 2, we begin with collecting some additional results from [BMM10].

The completion  $\overline{\mathcal{T}(S)}$  of  $\mathcal{T}(S)$  for the Weil-Petersson metric is a CAT(0)-space. Call two infinite Weil-Petersson geodesic rays  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$ ,  $\xi : [0, \infty) \rightarrow \mathcal{T}(S)$  *asymptotic* if the function

$$t \rightarrow d_{WP}(\gamma(t), \xi(t))$$

is bounded. Since  $\overline{\mathcal{T}(S)}$  is neither locally compact nor hyperbolic in the sense of Gromov (see [W03] for an overview and for references), it is difficult to find out whether or not for two given infinite non-asymptotic WP-rays  $\gamma_1, \gamma_2$  there is a biinfinite WP-geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .

Brock, Masur and Minsky [BMM10] found a sufficient condition for the existence of a biinfinite WP-geodesic which is forward and backward asymptotic to two given WP-rays. Namely, as in Section 2, call a WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  recurrent if there is a number  $\epsilon > 0$  and there is a sequence of numbers  $t_i \rightarrow \infty$  such that  $\gamma(t_i) \in \mathcal{T}(S)_\epsilon$  for all  $i$ . In other words, a geodesic ray is recurrent if its projection to moduli space returns to a fixed compact set for arbitrarily large times. Theorem 1.3 of [BMM10] shows that for every recurrent WP-geodesic ray  $\gamma$  and for *every* WP-geodesic ray  $\xi$  which is not asymptotic to  $\gamma$  there is a biinfinite WP-geodesic which is forward asymptotic to  $\gamma$  and backward asymptotic to  $\xi$ .

We use some ideas from [BMM10] to establish a related technical result (Corollary 3.2) which is used in an essential way in the proof of Theorem 1.

We begin with an observation which is a consequence of the Gauß-Bonnet formula for ruled surfaces as in [BMM10]. For its formulation, for  $\epsilon > 0$  let

$$(1) \quad 2b(\epsilon) = \inf\{d_{WP}(x, y) \mid x \in \mathcal{T}(S)_\epsilon, y \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)\}.$$

By invariance of  $\mathcal{T}(S)_\epsilon$  under the action of  $\text{Mod}(S)$  and cocompactness, we have  $b(\epsilon) > 0$  (in fact  $b(\epsilon) \asymp \epsilon^{1/2}$  by Wolpert's estimate [W03]). Moreover, the sectional curvature of the Weil-Petersson metric on the  $b(\epsilon)$ -neighborhood of  $\mathcal{T}(S)_\epsilon$  is bounded from above by a negative constant.

A *geodesic quadrangle* in  $(\mathcal{T}(S), d_{WP})$  consists of four WP-geodesic segments connecting four distinct points in  $\mathcal{T}(S)$ . We always assume that a geodesic quadrangle  $Q$  is non-degenerate, i.e. that no vertex of  $Q$  is contained in the interior of any side of  $Q$ . Two sides  $\alpha, \beta$  of such a quadrangle are *opposite* if they do not share a vertex.

For a Weil-Petersson geodesic segment  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  and  $\epsilon > 0$  let  $\ell_{\epsilon\text{-thick}}(\gamma)$  be the length of the intersection of  $\gamma$  with  $\mathcal{T}(S)_\epsilon$  (in other words,  $\ell_{\epsilon\text{-thick}}(\gamma)$  is the Lebesgue measure of the set  $\{t \in [0, \tau] \mid \gamma(t) \in \mathcal{T}(S)_\epsilon\}$ ).

In the remainder of this section, distances are always distances with respect to the Weil-Petersson metric. Moreover, angles are always unoriented angles with respect to the Weil-Petersson inner product.

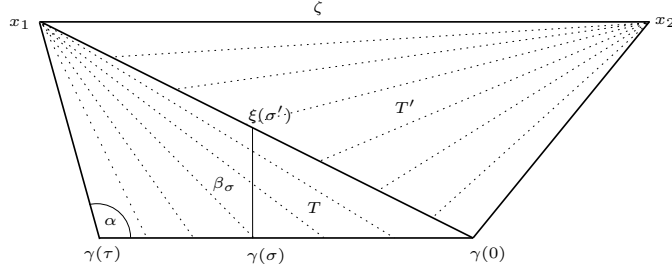
**Lemma 3.1.** (1) *For every  $\epsilon > 0$  and every  $\alpha > 0$  there is a number  $k_1 = k_1(\epsilon, \alpha) > 0$  with the following property. Let  $\tau \geq k_1$  and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma) \geq k_1$ . Assume that  $\gamma$  is a side of a geodesic quadrangle  $Q$  with angles at least  $\alpha$  at  $\gamma(0), \gamma(\tau)$ . Then the side of  $Q$  which is opposite to  $\gamma$  passes through the  $b(\epsilon)$ -neighborhood of  $\gamma[0, \tau] \cap \mathcal{T}(S)_\epsilon$ .*

(2) *For every  $\epsilon > 0, \alpha > 0$  and every  $\theta > 0$  there is a number  $k_2 = k_2(\epsilon, \alpha, \theta) > 0$  with the following property. Let  $\tau \geq k_2$  and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma) \geq k_2$ . Assume that  $\gamma$  is a side of a geodesic triangle  $T$  with angle at least  $\alpha$  at  $\gamma(\tau)$ . Then the angle of  $T$  at  $\gamma(0)$  is at most  $\theta$ .*

*Proof.* The idea of the proof is taken from [BMM10] (and has been used before by other authors, notably by Bonahon [Bo86] and Canary [C93]). Namely, let for the

moment  $\tau > 0$  be arbitrary and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment. Let  $Q$  be a WP-geodesic quadrangle with vertices  $\gamma(0), \gamma(\tau), x_1, x_2$  and such that  $\gamma(\tau)$  is connected to  $x_1$  by a side.

The vertices  $\gamma(0), \gamma(\tau), x_1$  determine a WP-geodesic triangle which can be filled by WP-geodesic segments issuing from the vertex  $x_1$  and connecting  $x_1$  to the opposite side  $\gamma$ . The thus obtained *ruled triangle*  $T$  is an embedded subsurface of  $\mathcal{T}(S)$  which is smooth in its interior, with piecewise geodesic boundary. The intrinsic distance in  $T$  between any two points  $x, y \in T$  is not smaller than  $d_{\text{WP}}(x, y)$ . Moreover, the (intrinsic) Gauß curvature of  $T$  with respect to the restriction of the Weil-Petersson metric at a point  $x \in T$  does not exceed the maximum of the sectional curvatures of the Weil-Petersson metric at  $x$ . In particular, the Gauß curvature of  $T$  is negative, and for every  $\epsilon > 0$  there is a constant  $\kappa(\epsilon) > 0$  only depending on  $\epsilon$  such that the Gauß curvature at every point in  $T$  whose distance to  $\mathcal{T}(S)_\epsilon$  is at most  $b(\epsilon)$  is bounded from above by  $-\kappa(\epsilon)$  (see [BMM10] for this construction of Bonahon [Bo86] and Canary [C93] and for references).



Even though the ruled triangle  $T$  is not naturally an embedded subsurface with piecewise geodesic boundary of a smooth simply connected surface  $U$  with a Riemannian metric, the intrinsic angle of  $T$  is defined at every vertex of  $T$ . At the vertex  $x_1$ , this angle is just the length of the arc in the unit tangent sphere for the Weil-Petersson metric which consists of the directions of all geodesics joining  $x_1$  to  $\gamma$ . This length exists since the initial direction of a geodesic depends smoothly on its endpoints. We claim that the intrinsic angles of  $T$  at the vertices  $\gamma(0), \gamma(\tau)$  coincide with the angles for the Weil-Petersson metric. Namely, by slightly extending the WP-geodesic  $\gamma$  and the geodesics defining the ruling of  $T$  we obtain a smooth subsurface of  $\mathcal{T}(S)$  containing a neighborhood of  $\gamma(0), \gamma(\tau)$  in its interior. The intrinsic angle at  $\gamma(0), \gamma(\tau)$  of the triangle  $T$  is just the angle of  $T$  at  $\gamma(0), \gamma(\tau)$  in this subsurface equipped with the restriction of the Weil-Petersson metric.

Let  $\epsilon > 0$  and let  $\kappa = \kappa(\epsilon) > 0$  be as in the second paragraph of this proof. Consider first the case that the angle of the quadrangle  $Q$  at the vertex  $\gamma(\tau)$  is not smaller than  $\pi/2$ . By the consideration in the previous paragraph, this angle coincides with the intrinsic angle at  $\gamma(\tau)$  of the triangle  $T$ . By the Gauß-Bonnet formula, the integral of the Gauß curvature over the triangle  $T$  equals the sum of the intrinsic angles of  $T$  minus  $\pi$ . Thus if  $\theta_1, \theta_2$  are the angles of  $T$  at  $\gamma(0), x_1$  then this curvature integral is not smaller than  $-\pi/2 + \theta_1 + \theta_2 > -\pi/2$ .

Denote by  $\xi$  be the side of  $T$  connecting  $\gamma(0) = \xi(0)$  to  $x_1$ . The triangle  $T$  is negatively curved and therefore intrinsic distance functions in  $T$  are convex. Since



the angle of  $T$  at  $\gamma(\tau)$  is not smaller than  $\pi/2$ , for each  $t \in [0, \tau]$  the endpoint of the intrinsic geodesic arc  $\beta_t$  in  $T$  which issues from  $\gamma(t)$  and is perpendicular to  $\gamma$  at  $\gamma(t)$  is contained in the side  $\xi$ . The length of  $\beta_t$  equals the intrinsic distance between its endpoint on  $\xi$  and the side  $\gamma$  of  $T$  and hence by convexity of the distance function, this length is increasing with  $t$ . Thus if there is a number  $\sigma \in [0, \tau]$  so that the length of  $\beta_\sigma$  is not smaller than  $b(\epsilon)/2$ , then for every  $s \in [\sigma, \tau]$  the length of  $\beta_s$  is not smaller than  $b(\epsilon)/2$ . Then  $T$  contains an embedded strip  $A_0$  of width  $b(\epsilon)/2$  and length  $\tau - \sigma$  which consists of all points in  $T$  on the initial subsegments of length  $b(\epsilon)/2$  of the geodesics  $\beta_s$  for all  $s \in [\sigma, \tau]$ .

Assume that there is such a point  $\sigma \in [0, \tau]$  such that moreover  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa(\epsilon)b(\epsilon)$  where  $\kappa(\epsilon) > 0$  is as in the second paragraph of this proof. Let  $A \subset A_0$  be the closed subset of the embedded strip  $A_0 \subset T$  which consists of the union of all initial subarcs of length  $b(\epsilon)/2$  of those of the geodesic arcs  $\beta_s$  which issue from a point in  $\gamma[\sigma, \tau] \cap \mathcal{T}(S)_\epsilon$ . Since  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa(\epsilon)b(\epsilon)$  and the curvature of  $T$  is negative, comparison with the euclidean plane shows that the area of  $A$  is at least  $\pi/2\kappa(\epsilon)$ . Now the Gauß curvature of  $T$  at every point of  $A$  does not exceed  $-\kappa(\epsilon)$  and therefore the integral of the Gauß curvature over  $T$  is smaller than  $-\pi/2$ . This contradicts the above observation that by the Gauß Bonnet formula, this curvature integral is bigger than  $-\pi/2$ . As a consequence, if  $\sigma \in [0, \tau]$  is such that  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa(\epsilon)b(\epsilon)$  then for  $s \in [0, \sigma]$  the length of the geodesic arc  $\beta_s$  does not exceed  $b(\epsilon)/2$ .

Now assume more restrictively that the angles of the quadrangle  $Q$  at the vertices  $\gamma(0), \gamma(\tau)$  are not smaller than  $3\pi/4$ . Then the angle of the triangle  $T$  at the vertex  $\gamma(\tau)$  is not smaller than  $3\pi/4$ . Since the triangle  $T$  is negatively curved, the sum of its angles is smaller than  $\pi$ . In particular, the Weil-Petersson angle at  $\gamma(0)$  between  $\gamma$  and the WP-geodesic  $\xi$  connecting  $\gamma(0)$  to  $x_1$  does not exceed  $\pi/4$ .

Assume that  $\ell_{\epsilon\text{-thick}}(\gamma) \geq 2\pi/\kappa(\epsilon)b(\epsilon)$  and let  $\sigma \in [0, \tau]$  be such that

$$\ell_{\epsilon\text{-thick}}(\gamma[0, \sigma]) \geq \pi/\kappa(\epsilon)b(\epsilon) \text{ and } \ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa(\epsilon)b(\epsilon).$$

Let  $\sigma' > \sigma$  be such that  $\xi(\sigma')$  is the endpoint of the geodesic segment  $\beta_\sigma$  in the ruled triangle  $T$  issuing perpendicularly from  $\gamma(\sigma)$ . Since  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa(\epsilon)b(\epsilon)$ , the above consideration shows that the distance between  $\gamma(\sigma)$  and  $\xi(\sigma')$  does not exceed  $b(\epsilon)/2$ . Since the distance function for the Weil-Petersson metric is convex, we conclude that  $d_{WP}(\xi(t), \gamma(t\sigma/\sigma')) \leq b(\epsilon)/2$  for all  $t \in [0, \sigma']$ .

Let  $T'$  be the ruled triangle obtained by connecting  $x_2$  to each point of  $\xi$  by a geodesic arc. The angle at  $\gamma(0)$  of  $T'$  is at least  $\pi/2$ . Apply the above discussion to the ruled triangle  $T'$ , the side  $\xi$  of  $T'$  and the side  $\zeta$  connecting  $x_1$  to  $x_2$ . Note that  $\zeta$  is the side of the quadrangle  $Q$  opposite to  $\gamma$ .

By the assumption that  $\ell_{\epsilon\text{-thick}}(\gamma[0, \sigma]) \geq \pi/\kappa(\epsilon)b(\epsilon)$ , if the distance between  $\zeta$  and  $\gamma[0, \sigma] \cap \mathcal{T}(S)_\epsilon$  is at least  $b(\epsilon)$  then there is an embedded strip in  $T'$  with curvature integral smaller than  $-\pi/2$ . This strip consists of all points on geodesic arcs in  $T'$  of length  $b(\epsilon)/2$  issuing perpendicularly from a point  $\xi(t)$  for some  $t \in [0, \sigma']$  such that  $\gamma(t\sigma/\sigma') \in \mathcal{T}(S)_\epsilon$ . Since the angle of  $T'$  at  $\gamma(0)$  is bounded from

below by  $\pi/2$ , this is impossible. As a consequence, the side  $\zeta$  intersects the  $b(\epsilon)$ -neighborhood of  $\gamma[0, \sigma] \cap \mathcal{T}(S)_\epsilon$ . This shows the first part of the lemma for  $\alpha = 3\pi/4$  with  $k_1(\epsilon, \alpha) = 2\pi/\kappa(\epsilon)b(\epsilon)$ .

The first part of the lemma for an arbitrary angle  $\alpha > 0$  is now a consequence of the second part of the lemma for  $\epsilon, \alpha$  and  $\theta = \pi/4$ .

Namely, assume that the second part of the lemma holds true and let  $Q$  be a geodesic quadrangle with a side  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  and angles at least  $\alpha$  at the vertices  $\gamma(0), \gamma(\tau)$ . Let  $x_1, x_2$  be the vertices of  $Q$  which are distinct from  $\gamma(0), \gamma(\tau)$  and assume that  $\ell_{\epsilon\text{-thick}}(\gamma) \geq 2\pi/\kappa(\epsilon)b(\epsilon) + 2k_2$  where  $k_2 = k_2(\epsilon, \alpha, \pi/4) > 0$  is as in the second part of the lemma. Let  $0 < t_1 < t_2 < \tau$  be such that

$$\ell_{\epsilon\text{-thick}}(\gamma[0, t_1]) \geq k_2, \ell_{\epsilon\text{-thick}}(\gamma[t_1, t_2]) \geq 2\pi/\kappa(\epsilon)b(\epsilon), \ell_{\epsilon\text{-thick}}(\gamma[t_2, \tau]) \geq k_2$$

and let  $Q'$  be the quadrangle with vertices  $\gamma(t_1), \gamma(t_2), x_1, x_2$ . The side  $\zeta$  of  $Q'$  opposite to  $\gamma[t_1, t_2]$  coincides with the side of  $Q$  opposite to  $\gamma$ . By the choice of  $t_1, t_2$  and by the second part of the lemma, applied to the triangle with vertices  $\gamma(0), \gamma(t_1), x_2$  and an angle at least  $\alpha$  at  $\gamma(0)$  and the triangle with vertices  $\gamma(t_2), \gamma(\tau), x_1$  and an angle at least  $\alpha$  at  $\gamma(\tau)$ , the angle of  $Q'$  at the vertices  $\gamma(t_1), \gamma(t_2)$  is at least  $3\pi/4$ . Therefore we conclude from the first part of the lemma for  $\alpha = 3\pi/4$  that  $\zeta$  passes through the  $b(\epsilon)$ -neighborhood of  $\gamma[t_1, t_2] \cap \mathcal{T}(S)_\epsilon$ . This is what we wanted to show.

To establish the angle estimate in the second part of the lemma, let  $\alpha > 0, \theta > 0$  and let  $T$  be a triangle in  $(\mathcal{T}(S), d_{WP})$  with a side  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  of length  $\sigma > 0$ , an angle  $\theta_0 \geq \theta$  at  $\zeta(0)$  and an angle  $\alpha_0 \geq \alpha$  at  $\zeta(\sigma)$ . Let  $x_1$  be the vertex of  $T$  opposite to the side  $\zeta$  and assume that  $T$  is ruled by WP-geodesics connecting  $x_1$  to the points on  $\zeta$ . The Gauss curvature of  $T$  is negative, in particular we have  $\alpha_0 + \theta_0 < \pi$ .

Let  $\hat{T}$  be a comparison triangle in the euclidean plane  $\mathbb{R}^2$  with a side  $\hat{\zeta} : [0, \sigma] \rightarrow \mathbb{R}^2$  of length  $\sigma$ , an angle  $\theta$  at  $\hat{\zeta}(0)$  and an angle  $\alpha$  at  $\hat{\zeta}(\sigma)$ . Note that the angles of  $\hat{T}$  at  $\hat{\zeta}(0), \hat{\zeta}(\sigma)$  may both be smaller than the corresponding angles of  $T$ . Since  $\theta_0 \geq \theta$ , by CAT(0)-comparison (see [BH99]), for every  $s \in [0, \sigma]$  the length of the intrinsic geodesic  $\beta_s$  in the ruled triangle  $T$  which is orthogonal to  $\zeta$  at  $\zeta(s)$  and which ends on one of the two sides different from  $\zeta$  is not smaller than the length of the geodesic  $\hat{\beta}_s$  in  $\hat{T}$  which is orthogonal to  $\hat{\zeta}$  at  $\hat{\zeta}(s)$  and ends on one of the sides of  $\hat{T}$  distinct from  $\hat{\zeta}$ .

As the length  $\sigma$  of  $\hat{\zeta}$  tends to infinity, the distance between  $\hat{\zeta}$  and the vertex of  $\hat{T}$  not contained on  $\hat{\sigma}$  tends to infinity as well. Thus there is a number  $t(\alpha, \theta) > 0$  only depending on  $\alpha$  and  $\theta$  such that for  $s \in [t(\alpha, \theta), \sigma - t(\alpha, \theta)]$  the length of the geodesic arc  $\beta_s$  in  $T$  is not smaller than  $b(\epsilon)/2$ . In particular, by comparison, the ruled triangle  $T$  contains an embedded strip of area at least  $b(\epsilon)(\sigma - 2t(\alpha, \theta))/2$  which is the union of the initial subsegments of length  $b(\epsilon)/2$  of the geodesic arcs  $\beta_s$  issuing from points  $\zeta(s)$  where  $s \in [t(\alpha, \theta), \sigma - t(\alpha, \theta)]$ .

Now by the argument in the beginning of this proof, if  $\rho = \ell_{\epsilon\text{-thick}}(\zeta)$  then the integral of the Gauss curvature over this strip is at most  $-\kappa(\epsilon)b(\epsilon)(\rho - 2t(\alpha, \theta))/2$ , and hence the sum of the angles of the triangle  $T$  is at most  $\pi - \kappa(\epsilon)b(\epsilon)(\rho -$

$2t(\alpha, \theta))/2$ . On the other hand, the angle sum of  $T$  is at least  $\alpha + \theta$  by assumption which implies that

$$\rho \leq 2(\pi - \alpha - \theta)/\kappa(\epsilon)b(\epsilon) + 2t(\alpha, \theta).$$

As a consequence, if  $T$  is a WP-geodesic triangle in  $\mathcal{T}(S)$  with a side  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  such that  $\ell_{\epsilon\text{-thick}}(\zeta) \geq 2(\pi - \alpha - \theta)/\kappa(\epsilon)b(\epsilon) + 2t(\alpha, \theta)$  and if the angle of  $T$  at  $\zeta(\sigma)$  is not smaller than  $\alpha$ , then the angle of  $T$  at  $\zeta(0)$  does not exceed  $\theta$ . This is just the statement in the second part of the lemma.  $\square$

As a consequence of Lemma 3.1, we obtain a sufficient condition for the existence of a biinfinite WP-geodesic which is forward and backward asymptotic to WP-rays with specific geometric properties which does not use require these rays to be recurrent.

**Corollary 3.2.** *For  $\epsilon > 0$  and  $\alpha > 0$  let  $k_1 = k_1(\epsilon, \alpha) > 0$  be as in Lemma 3.1. Let  $\gamma_0 : [0, \tau] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma_0) \geq k_1$  and let  $\gamma_1, \gamma_2 : [0, \infty) \rightarrow \mathcal{T}(S)$  be infinite WP-rays issuing from  $\gamma_1(0) = \gamma_0(0), \gamma_2(0) = \gamma_0(\tau)$ . Assume that the angle at  $\gamma_0(0), \gamma_0(\tau)$  between the unit tangent vectors  $\gamma'_0(0), \gamma'_1(0)$  and between the unit tangent vectors  $-\gamma'_0(\tau), \gamma'_2(0)$  is at least  $\alpha$ . Assume furthermore that there are measured geodesic laminations  $\mu_1, \mu_2$  which fill  $S$  and such that the length of  $\mu_i$  is bounded along  $\gamma_i$  ( $i=1,2$ ). Then there is a unique biinfinite WP-geodesic  $\xi$  which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .*

*Proof.* Let  $\gamma_0, \gamma_1, \gamma_2$  be as in the lemma. For  $t > 0$  consider the geodesic quadrangle  $Q_t$  with vertices  $\gamma_0(0), \gamma_0(\tau), \gamma_2(t), \gamma_1(t)$ . By Lemma 3.1, the side  $\xi_t$  of  $Q_t$  which connects  $\gamma_1(t)$  to  $\gamma_2(t)$  passes through a fixed compact neighborhood  $K$  of  $\gamma_0[0, \tau] \cap \mathcal{T}(S)_\epsilon$ . We parametrize  $\xi_t$  by arc length in such a way that  $\xi_t(0) \in K$ . Then the WP-distance between  $\gamma_0(0)$  and  $\xi_t(0)$  is uniformly bounded, independent of  $t$ .

By the CAT(0)-triangle comparison property, the Hausdorff distance with respect to the Weil-Petersson metric between the geodesic arc  $\gamma_1[0, t]$  and the subsegment of  $\xi_t$  connecting  $\xi_t(0)$  to  $\gamma_1(t)$  is uniformly bounded, independent of  $t$ . Similarly, the Hausdorff distance with respect to the Weil-Petersson metric between the geodesic arc  $\gamma_2[0, t]$  and the subsegment of  $\xi_t$  connecting  $\xi_t(0)$  to  $\gamma_2(t)$  is uniformly bounded.

The restriction to  $K$  of the unit tangent bundle for the Weil-Petersson metric is compact. Therefore there is a sequence  $t_i \rightarrow \infty$  such that the directions  $\xi'_{t_i}(0)$  of  $\xi_{t_i}$  at  $\xi_{t_i}(0)$  converge as  $i \rightarrow \infty$  to a direction  $v$ . Let  $\xi : (-r, T) \rightarrow \mathcal{T}(S)$  be the (maximal) WP-geodesic with initial velocity  $v$ . The WP-geodesics  $\xi_{t_i}$  converge uniformly on compact subsets of  $(-r, T)$  to  $\xi$ . In particular, if  $\xi$  is biinfinite (i.e. if  $r = T = \infty$ ) then  $\xi$  is indeed a geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .

To see that  $\xi$  is indeed biinfinite, let  $\mu_1 \in \mathcal{ML}$  be a measured geodesic lamination which fills up  $S$  and whose length is bounded along the geodesic ray  $\gamma_1$ . Such a measured geodesic lamination exists by assumption. Since  $\xi_t(0) \in K$  for all  $t$ , the  $\xi_t(0)$ -length of  $\mu_1$  is bounded independent of  $t > 0$ . Now for every  $t$  the forward endpoint  $\xi_t(\tau_t)$  of  $\xi_t$  equals  $\gamma_1(t)$  and hence by convexity of length functions along

WP-geodesics, the length of  $\mu_1$  is bounded along  $\xi_t[0, \tau_t]$  by a universal constant, independent of  $t$ . Note also that  $\tau_t \rightarrow \infty$  ( $t \rightarrow \infty$ ). By continuity of the length pairing, we conclude that the length of  $\mu_1$  on  $\xi[0, T)$  is uniformly bounded (compare the proof of Lemma 2.7 and Proposition 2.8 for a more detailed argument). However, if  $T < \infty$  then there is a simple closed curve  $c$  on  $S$  so that the  $\xi(t)$ -length of  $c$  tends to zero as  $t \rightarrow T$  (see [W03] for more and for references). Since  $\mu_1$  fills  $S$  we have  $i(c, \mu_1) > 0$  and therefore the length of  $\mu_1$  along  $\xi[0, T)$  tends to infinity as  $t \rightarrow T$  which is a contradiction. This argument also applies to the ray  $\xi(-r, 0]$  and yields that  $\xi$  is indeed a biinfinite WP-geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .

To show that such a geodesic is unique, recall from comparison for CAT(0)-spaces that if there is a second such geodesic  $\xi'$  then  $\xi$  and  $\xi'$  bound a flat strip. Since the sectional curvature of the Weil-Petersson metric is negative, this is impossible. The corollary follows.  $\square$

#### 4. SHORT CURVES AND TWISTING

In Corollary 3.2 we established a sufficient condition for the existence of a biinfinite Weil-Petersson geodesic which is forward and backward asymptotic to two given Weil-Petersson geodesic rays. To apply this result, we have to find a sufficient condition for a Weil-Petersson geodesic to spend a definitive amount of time in the thick part of Teichmüller space.

We approach this problem by analyzing WP-geodesic segments of uniformly bounded length which enter deeply into the thin part of Teichmüller space. Our goal is a quantitative version of the following result of Wolpert [W03]: If  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  is any Weil-Petersson geodesic of uniformly bounded length and if  $\gamma$  enters the thin part of Teichmüller space then  $\gamma$  twists a definitive amount about one of the curves which becomes very short along  $\gamma$ . Or, put differently, if  $\gamma$  does not twist much about its short curves then  $\gamma$  can not be entirely contained in the thin part of Teichmüller space.

We continue to use the assumptions and notations from Sections 2-3. Following Masur and Minsky, we measure the amount of twisting about a curve  $\alpha \in \mathcal{C}(S)$  as follows (see in particular p. 919 of [MM00]). Fix a complete hyperbolic metric of finite volume  $h \in \mathcal{T}(S)$  and identify  $\alpha$  with the  $h$ -geodesic it defines. There is a locally isometric annular cover  $\tilde{A} \rightarrow S$  of  $(S, h)$  whose geodesic core curve  $\tilde{\alpha}$  projects isometrically onto  $\alpha$ . The hyperbolic annulus  $\tilde{A}$  admits a natural compactification to a closed annulus  $\hat{A}$  which is obtained as follows. The fundamental group  $\langle \alpha \rangle$  of  $\tilde{A}$  acts on the hyperbolic plane  $\mathbf{H}^2$  as a group of hyperbolic isometries fixing two points  $a \neq b$  in the ideal boundary  $\partial \mathbf{H}^2$  of  $\mathbf{H}^2$ . The quotient of  $\mathbf{H}^2 \cup (\partial \mathbf{H}^2 - \{a, b\})$  under the action of  $\langle \alpha \rangle$  is a compact annulus  $\hat{A}$  containing  $\tilde{A}$  as an open dense subset. Any geodesic in  $\tilde{A}$  for the hyperbolic metric which intersects the geodesic core curve  $\tilde{\alpha}$  transversely extends to a continuous path in  $\hat{A}$  connecting the two distinct boundary components of  $\hat{A}$ .

Let  $\mathcal{C}(\alpha)$  be the set of all simple paths in  $\hat{A}$  connecting the two distinct boundary components of  $\hat{A}$  modulo homotopies that fix the endpoints. Then  $\mathcal{C}(\alpha)$  is the set of vertices of a metric graph  $\mathcal{CG}(\alpha)$  whose edges are determined by requiring that two such homotopy classes of arcs  $\gamma, \gamma'$  are connected by an edge of length one if and only if  $\gamma, \gamma'$  have representatives with disjoint interior.

Following p. 920 of [MM00], define a projection  $\pi_\alpha$  of  $\mathcal{C}(S)$  into the family of all subsets of  $\mathcal{CG}(\alpha)$  as follows. Let  $\gamma \in \mathcal{C}(S)$  be represented by a simple closed  $h$ -geodesic. If  $\gamma$  does not intersect  $\alpha$  transversely then we define  $\pi_\alpha(\gamma) = \emptyset$ . Otherwise the preimage  $\tilde{\gamma}$  of  $\gamma$  in  $\tilde{A}$  has at least one component which extends continuously to an arc connecting the two distinct boundary components of  $\hat{A}$ . The set  $\pi_\alpha(\gamma)$  of all these components is a finite set of diameter at most 1 in  $\mathcal{CG}(\alpha)$ . This definition of  $\pi_\alpha$  is essentially independent of the choice of the hyperbolic metric  $h$  (see [MM00] for more details). If  $c$  is a *simple multi-curve*, i.e. a disjoint union of mutually not freely homotopic simple closed curves, then let  $\pi_\alpha(c)$  be the union of the projections of its components. As before, the diameter of  $\pi_\alpha(c)$  is at most one (Lemma 2.3 of [MM00]). The projection  $\pi_\alpha$  can be used to measure the relative twisting about  $\alpha$  of two simple closed curves which intersect  $\alpha$  transversely [MM00].

As in Section 2, let  $\chi_0 > 0$  be a Bers constant for  $S$  and let  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$  be a map which associates to a hyperbolic metric  $h$  a simple closed curve of  $h$ -length at most  $\chi_0$ . We use the projections  $\pi_\alpha$  ( $\alpha \in \mathcal{C}(S)$ ) and Wolpert's description of the Weil-Petersson metric near its completion locus (see [W03, W08]) to obtain information on the image of a Weil-Petersson geodesic  $\gamma$  under the map  $\Upsilon_{\mathcal{T}}$ . We are in particular interested in the twisting behavior of points in  $\Upsilon_{\mathcal{T}}(\gamma)$  about a simple closed curve which becomes very short along  $\gamma$ .

For this we first need a better control of the projections of WP-geodesics into the graph  $\mathcal{CG}(\alpha)$  for a simple closed curve  $\alpha \in \mathcal{C}(S)$ . We obtain such a control using the *pants graph*  $\mathcal{PG}(S)$  for  $S$ .

A pants decomposition  $P$  for  $S$  is changed to a pants decomposition  $P'$  by an *elementary move* if  $P'$  is obtained from  $P$  by replacing one of the pants curves  $\alpha$  of  $P$  by a curve which does not intersect  $P - \alpha$  and intersects  $\alpha$  in the minimal number of points (i.e. in precisely two points if the component of  $S - (P - \alpha)$  containing  $\alpha$  is a four-holed sphere, and in precisely one point if this component is a one-holed torus). The pants graph  $\mathcal{PG}(S)$  of  $S$  is the geodesic metric graph whose set of vertices is the set  $\mathcal{P}(S)$  of pants decompositions for  $S$  and where two such pants decompositions  $P, P'$  are connected by an edge of length one if and only if  $P'$  can be obtained from  $P$  by an elementary move.

As in Section 2, call a pants decomposition  $P$  for  $S$  a Bers decomposition for  $h \in \mathcal{T}(S)$  if the  $h$ -length of each of the components of  $P$  is bounded from above by  $\chi_0$ . Note that if  $\alpha \in \mathcal{C}(S)$  is any simple closed curve whose  $h$ -length is bigger than  $\chi_0$  then every Bers decomposition  $P$  for  $h$  contains a component which intersects  $\alpha$  transversely. In particular, the projection  $\pi_\alpha(P)$  is not empty, and its diameter  $\text{diam}(\pi_\alpha(P))$  is at most one.

Define a map

$$\Upsilon_{\mathcal{P}} : \mathcal{T}(S) \rightarrow \mathcal{PG}(S)$$

by associating to a hyperbolic metric  $x \in \mathcal{T}(S)$  a Bers decomposition for  $x$ . The following Lemma is a quantitative version of Lemma 2.4 for pants decompositions which enables us to control for a simple closed curve  $\alpha$  the projections into the graph  $\mathcal{CG}(\alpha)$  of Bers decompositions along Weil-Petersson geodesics. Compare also Section 3 of [B03].

**Lemma 4.1.** *There is a constant  $\chi_1 > \chi_0$  with the following property. Let  $\xi : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment of length  $\sigma \leq 1$  and let  $P_0, P_\sigma$  be Bers decompositions for  $\xi(0), \xi(\sigma)$ . Then  $P_0$  can be connected to  $P_\sigma$  by an edge path  $\rho$  in  $\mathcal{PG}(S)$  of length at most  $\chi_1$  with the following property. For every vertex  $P \in \mathcal{P}(S)$  passed through by  $\rho$  there is some  $t \in [0, \sigma]$  such that the  $\xi(t)$ -length of every component curve of  $P$  is smaller than  $\chi_1$ .*

*Proof.* By Lemma 3.12 of [W08], there is a number  $a > 0$  with the following property. Let  $h \in \mathcal{T}(S)$  and let  $\alpha \in \mathcal{C}(S)$  be a simple closed curve of  $h$ -length  $\ell_h(\alpha) \leq 2\chi_0$ . Then the norm at  $h$  of the Weil-Petersson gradient of the length function  $x \rightarrow \ell_x(\alpha)$  is bounded from above by  $a$ . This implies the following. Let  $\tau \leq \chi_0/a$ , let  $\xi : [0, \tau] \rightarrow \mathcal{T}(S)$  be any WP-geodesic and let  $P$  be a Bers decomposition for  $\xi(0)$ . Then for every  $t \in [0, \tau]$  the  $\xi(t)$ -length of every component of  $P$  does not exceed  $2\chi_0$ .

Choose an integer  $\ell > a/\chi_0$  and note that  $\ell$  is a universal constant. Let  $\xi : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic of length  $\sigma \leq 1$  and let  $j \leq \ell$  be the smallest integer such that  $j/\ell \geq \sigma$ . Let  $P_0, P_\sigma$  be Bers decompositions for  $\xi(0), \xi(\sigma)$  and for each  $i \in \{1, \dots, j-1\}$  let  $P_i$  be a Bers decomposition for  $\xi(\frac{i}{\ell})$ . By the choice of the constant  $\ell$ , for each  $i$  the  $\xi(\frac{i+1}{\ell})$ -length of each component of  $P_i$  does not exceed  $2\chi_0$ . Since  $j \leq \ell$ , for the proof of the lemma it is enough to show the existence of a number  $\beta > 0$  with the following property. For every  $x \in \mathcal{T}(S)$ , any pants decomposition  $Q_0$  of  $S$  with components of  $x$ -length at most  $2\chi_0$  can be connected to a given Bers decomposition  $Q_1$  for  $x$  by a path in  $\mathcal{PG}(S)$  of length at most  $\beta$  passing through vertices of the pants graph which are pants decompositions for  $S$  with components of  $x$ -length at most  $\beta$ .

Thus let  $x \in \mathcal{T}(S)$  and let  $Q_0, Q_1$  be such pants decompositions whose components are  $x$ -geodesics of length at most  $2\chi_0$  and  $\chi_0$ , respectively. Let  $c_1, \dots, c_k$  ( $0 \leq k \leq 3g - 3 + m$ ) be those components of  $Q_0$  which are also components of  $Q_1$ . Note that by the collar lemma, the set  $c_1, \dots, c_k$  contains every simple closed  $x$ -geodesic of sufficiently small length. Let  $\hat{S}$  be the metric completion of the (perhaps disconnected) surface which we obtain by cutting  $S$  open along the geodesics  $c_1, \dots, c_k$ . Choose a component  $\hat{S}_0$  of  $\hat{S}$  which is different from a three-holed sphere.

The intersection  $Q_0 \cap \hat{S}_0$  of  $Q_0$  with the interior of  $\hat{S}_0$  is a pants decomposition for  $\hat{S}_0$ , and the same is true for the intersection  $Q_1 \cap \hat{S}_0$  of  $Q_1$  with the interior of  $\hat{S}_0$ . Thus  $(Q_0 \cup Q_1) \cap \hat{S}_0$  is an embedded connected piecewise geodesic graph  $G$  in  $\hat{S}_0$  which decomposes  $\hat{S}_0$  into non-essential annuli, i.e. annuli whose core curves are homotopic to a boundary component of  $\hat{S}_0$  or to a puncture, and into topological discs with piecewise geodesic boundary. In particular, the injection of the graph  $G$  into  $\hat{S}_0$  induces a surjection of fundamental groups. By the collar lemma and the length bound for the components of  $Q_0, Q_1$ , the number of intersections between

$Q_0$  and  $Q_1$  is bounded from above by a number only depending on  $\chi_0$  and the topological type of  $S$ . This number of intersections is the number of vertices of the graph  $G$ . Now the valency of a vertex of  $G$  equals four and therefore the number of edges of  $G$  is uniformly bounded as well. The length of each edge does not exceed  $2\chi_0$ .

Up to the action of the mapping class group  $\text{Mod}(\hat{S}_0)$  of  $\hat{S}_0$ , there are only finitely many pairs of pants decompositions of  $\hat{S}_0$  whose union is an embedded connected graph in  $\hat{S}_0$  with a uniformly bounded number of edges and whose complementary components are topological discs and non-essential annuli. By invariance under the action of  $\text{Mod}(\hat{S}_0)$ , this means that there is a number  $p = p(\hat{S}_0) > 0$  only depending on the topological type of  $\hat{S}_0$ , there is a number  $n \leq p$  and there is a sequence of pants decompositions  $R_0 = Q_0 \cap \hat{S}_0, R_1, \dots, R_n = Q_1 \cap \hat{S}_0$  for  $\hat{S}_0$  with the following properties. For each  $i < n$ ,  $R_{i+1}$  can be obtained from  $R_i$  by an elementary move. Moreover, each simple closed curve on  $\hat{S}_0$  appearing as a pants curve of one of the pants decompositions  $R_i$  is freely homotopic to an edge path in the graph  $(Q_0 \cup Q_1) \cap \hat{S}_0$  of uniformly bounded combinatorial length. Since the  $x$ -length of an edge of  $(Q_0 \cup Q_1) \cap \hat{S}_0$  is uniformly bounded, the  $x$ -length of each component curve of  $R_i$  is bounded from above by a constant which only depends on  $\hat{S}_0$ .

On the other hand, there are only finitely many topological types of subsurfaces of  $S$  which can arise as complementary components of a simple multi-curve on  $S$ . Thus a successive application of this construction to all components of  $\hat{S}$  shows that  $Q_0$  can be modified to  $Q_1$  in a uniformly bounded number of steps consisting of pants decompositions whose components have uniformly bounded  $x$ -length. The lemma is proven.  $\square$

Lemma 4.1 together with the results of [MM00] imply the following projection-diameter control.

**Corollary 4.2.** *Let  $\xi : [0, R] \rightarrow \mathcal{T}(S)$  be any Weil-Petersson geodesic. Let  $\alpha \in \mathcal{C}(S)$  be a simple closed curve with  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for every  $t \in [0, R]$ , where  $\chi_1 > 0$  is as in Lemma 4.1. Let  $P, Q$  be Bers decompositions for  $\xi(0), \xi(R)$ . Then*

$$\text{diam}(\pi_\alpha(P) \cup \pi_\alpha(Q)) \leq 4\chi_1 R + 8\chi_1.$$

*Proof.* Let  $\xi : [0, R] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic and let  $P_0, P_1$  be Bers decompositions for  $\xi(0), \xi(R)$ . Let moreover  $\alpha \in \mathcal{C}(S)$  be a simple closed curve with  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for every  $t \in [0, R]$ . By Lemma 4.1,  $P_0$  can be connected to  $P_1$  by a path in  $\mathcal{PG}(S)$  of length  $n \leq R\chi_1 + \chi_1$  which passes through vertices  $Q_0 = P_0, \dots, Q_n = P_1$  of the pants graph defined by pants decompositions with component curves of  $\xi(t)$ -length smaller than  $\chi_1$  for some  $t \in [0, R]$ . Since  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for all  $t$ , each of the pants decompositions  $Q_i$  intersects  $\alpha$  transversely.

By Lemma 2.3 of [MM00], for every pants decomposition  $P$  of  $S$  with an essential intersection with  $\alpha$ , the projection  $\pi_\alpha(P)$  of  $P$  to  $\mathcal{CG}(\alpha)$  is non-empty and of diameter at most 2. If for some  $i < n$  the pants decomposition  $Q_{i+1}$  is obtained from  $Q_i$  by an elementary move preserving at least one curve which intersects  $\alpha$  transversely,

then the projection of this curve to  $\mathcal{CG}(\alpha)$  is contained in  $\pi_\alpha(Q_i) \cap \pi_\alpha(Q_{i+1})$  and therefore the diameter of  $\pi_\alpha(Q_i) \cup \pi_\alpha(Q_{i+1})$  is at most 4. Otherwise the elementary move which transforms  $Q_i$  to  $Q_{i+1}$  exchanges two simple closed curves  $\beta_i, \beta_{i+1}$ , and the connected component  $Y$  of  $S - (Q_i - \beta_i)$  distinct from a pair of pants contains  $\alpha$ . Now  $Y$  is a four-holed sphere or a one-holed torus which is bounded by simple closed curves in  $Q_i \cap Q_{i+1}$ , and  $\alpha$  intersects both  $\beta_i$  and  $\beta_{i+1}$  transversely. However, in this case the diameter of  $\pi_\alpha(Q_i) \cup \pi_\alpha(Q_{i+1}) = \pi_\alpha(\beta_i) \cup \pi_\alpha(\beta_{i+1})$  is also bounded from above by 4 by another application of Lemma 2.3 of [MM00].

As a consequence and by induction, the diameter in  $\mathcal{CG}(\alpha)$  of the projection  $\pi_\alpha(P_0) \cup \pi_\alpha(P_1)$  is at most  $4(n+1) \leq 4\chi_1(R+1) + 4 \leq 4\chi_1(R+1) + 8\chi_1$  (note that  $\chi_1 > 1$  by construction). The corollary follows.  $\square$

Let again  $\chi_0 > 0$  be a Bers constant for  $S$  and let  $\chi_1 > \chi_0$  be as in Lemma 4.1. By the collar lemma and the fact that the distance in  $\mathcal{CG}(S)$  between any two simple closed curves  $\alpha, \beta \in \mathcal{C}(S)$  does not exceed  $i(\alpha, \beta) + 1$  (Lemma 2.1 of [MM99]), there is a number  $p = p(\chi_0, \chi_1) > 0$  with the following property. Let  $x \in \mathcal{T}(S)$  and let  $\alpha$  be a simple closed curve on  $S$  whose  $x$ -length is at most  $\chi_0$ . If  $d_{\mathcal{C}}(\alpha, \beta) \geq p - 1$  then the  $x$ -length of  $\beta$  is bigger than  $\chi_1$ .

We use Corollary 4.2 and the results of Wolpert [W03] to control Weil-Petersson geodesics in the thin part of Teichmüller space.

**Proposition 4.3.** *For every  $R > 1, c > 0$  there is a number  $\epsilon = \epsilon(R, c) > 0$  with the following property. Let  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment of length  $\sigma \leq R$  and let  $P_0, P_\sigma$  be Bers decompositions for  $\zeta(0), \zeta(\sigma)$ . Let  $\alpha \in \mathcal{C}(S)$  be a curve whose distance in  $\mathcal{CG}(S)$  to any component of  $P_0, P_\sigma$  is at least  $p$ . Assume that  $\text{diam}(\pi_\beta(P_0) \cup \pi_\beta(P_\sigma)) \leq c$  for every simple closed curve  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha, \beta) \leq 1$ . Then  $\ell_{\zeta(t)}(\alpha) \geq \epsilon$  for all  $t \in [0, \sigma]$ .*

*Proof.* The proof relies on Wolpert's description of Weil-Petersson geodesics near the completion locus of Teichmüller space as explained in [W03].

We argue by contradiction and we assume that the statement of the proposition does not hold. Then there are numbers  $R > 1, c > 0$  and there is a sequence  $\epsilon_i \rightarrow 0$ , a sequence of WP-geodesics  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$ , a sequence of numbers  $t_i \in (0, \sigma_i)$  and a sequence of simple closed curves  $\alpha_i \in \mathcal{C}(S)$  such that for every  $i$  the following holds true.

- (a)  $\sigma_i \leq R$ .
- (b) The distance in  $\mathcal{CG}(S)$  between  $\alpha_i$  and any component of some Bers decomposition  $P_i, P_{\sigma_i}$  of  $\zeta_i(0), \zeta_i(\sigma_i)$  is at least  $p$ .
- (c)  $\text{diam}(\pi_\beta(P_i) \cup \pi_\beta(P_{\sigma_i})) \leq c$  for every simple closed curve  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha_i, \beta) \leq 1$ .
- (d) There is some  $t_i \in (0, \sigma_i)$  such that  $\ell_{\zeta_i(t_i)}(\alpha_i) \leq \epsilon_i$ .



Our strategy is to analyze a sequence of geodesics  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$  which has the properties (a), (b) and (d) above. By the choice of the constants  $p > 0$  and  $\chi_1 > 0$ , for each  $i$  we have

$$(2) \quad \ell_{\zeta_i(0)}(\alpha_i) \geq \chi_1, \ell_{\zeta_i(\sigma_i)}(\alpha_i) \geq \chi_1.$$

A result of Wolpert [W03] gives some geometric information on the sequence. This information allows us to formulate an additional condition on the sequence. We then consider sequences which satisfy this additional condition and use Corollary 4.2 to show that for such a sequence, property (c) above is violated. The general case is then reduced to the special case, applied to subarcs of the sequence  $\zeta_i$ .

Let now  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$  be a sequence with properties (a),(b),(d). By Wolpert's gradient estimates for length functions (see Lemma 3.12 of [W08]), for every  $\beta \in \mathcal{C}(S)$  the norm of the Weil-Petersson gradient of the length function  $x \rightarrow \ell_\beta(x)$  is uniformly bounded on  $\{x \mid \ell_\beta(x) \leq \chi_1\}$ . This implies that the Weil-Petersson distance between a point  $x \in \mathcal{T}(S)$  with  $\ell_{\alpha_i}(x) \geq \chi_1$  and a point  $y \in \mathcal{T}(S)$  with  $\ell_{\alpha_i}(y) \leq \chi_0/2$  is bounded from below by a universal constant  $a > 0$ . In particular, we have  $\sigma_i \geq 2a$  for all  $i$  which are sufficiently large that  $\epsilon_i \leq \chi_0/2$  and hence by passing to a subsequence we may assume that  $\sigma_i \rightarrow \sigma \in [2a, R]$ .

The mapping class group acts on the Weil-Petersson completion  $\overline{\mathcal{T}(S)}$  of Teichmüller space. The quotient  $\overline{\mathcal{T}(S)}/\text{Mod}(S)$  is just the Deligne-Mumford compactification of moduli space, in particular it is compact. Thus up to passing to another subsequence and up to the action of the mapping class group, we may assume that the initial points  $\zeta_i(0)$  of the geodesics  $\zeta_i$  converge to a point  $x_0 \in \overline{\mathcal{T}(S)}$  (compare the discussion in [W03]).

A point in  $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$  is a surface with nodes, where a node is obtained by pinching a simple closed curve on  $S$  to a point. For a simple multi-curve  $c$  on  $S$  let  $\mathcal{T}(c) \subset (\overline{\mathcal{T}(S)} - \mathcal{T}(S))$  be the stratum of the completion locus for the Weil-Petersson metric which consists of all Riemann surfaces with nodes at the components of  $c$  (i.e. Riemann surfaces obtained from  $S$  by pinching the components of  $c$  to punctures). By Proposition 23 of [W03], up to passing to another subsequence and up to possibly a composition with Dehn multi-twists about the nodes of  $x_0$ , there exists a finite partition  $0 = t_0 < t_1 < \dots < t_k = \sigma$  of the interval  $[0, \sigma]$  and there are simple multi-curves  $c_0, c_1, \dots, c_k$  and points  $x_j \in \mathcal{T}(c_j)$  such that the following holds true.

For  $0 \leq j \leq k-1$  consider the (possibly trivial) multi-curve  $\tau_j = c_j \cap c_{j+1}$ . If  $1 \leq j < k-1$  then  $\tau_j$  is a proper subset of both  $c_j$  and  $c_{j+1}$ , and  $\tau_0 = c_0 \cap c_1$  is a proper subset of  $c_1$ ,  $\tau_{k-1}$  is a proper subset of  $c_{k-1}$ . For each  $i$  and each  $1 \leq j \leq k-1$  there is a Dehn multi-twist  $T_{(j,i)}$  about the components of  $c_j - \tau_j$  such that on the parameter interval  $[t_j, t_{j+1}]$  the arcs

$$(3) \quad T_{(j,i)} \circ \dots \circ T_{(1,i)} \zeta_i$$

converge as  $i \rightarrow \infty$  to the geodesic arc  $\xi_j$  in  $\overline{\mathcal{T}(S)}$  connecting  $x_j$  to  $x_{j+1}$  in the sense of parametrized unit-speed curves. The concatenation  $\xi : [0, \sigma] \rightarrow \overline{\mathcal{T}(S)}$  of the arcs  $\xi_j$  ( $0 \leq j \leq k-1$ ) is the piecewise Weil-Petersson geodesic connecting  $x_0$  to  $x_k$  and passing through  $x_1, \dots, x_{k-1}$  in this order.

For each  $i$  the  $\zeta_i(0)$ -length and the  $\zeta_i(\sigma_i)$ -length of the curve  $\alpha_i$  is bounded from below by  $\chi_1$ , and the minimum of the length of  $\alpha_i$  along the WP-geodesics  $\zeta_i$  tends to zero as  $i \rightarrow \infty$ . This implies that  $k \geq 2$  and that up to passing to a subsequence, there is a number  $j_2 \in \{1, \dots, k-1\}$  such that for every sufficiently large  $i$  we have

$$T_{(j_2-1,i)} \circ \dots \circ T_{(1,i)} \alpha_i = \alpha \in c_{j_2} - \tau_{j_2}.$$

By Proposition 23 of [W03], the sequence of Dehn multi-twists  $T_{(j_2,i)}$  about the components of  $c_{j_2} - \tau_{j_2}$  is unbounded as  $i \rightarrow \infty$ . In particular, up to passing to a subsequence there is a simple closed curve  $\beta \in c_{j_2} - \tau_{j_2}$  with the following property. Let  $T_\beta$  be the Dehn twist about  $\beta$ . Then there is a sequence  $r(i) \rightarrow \infty$  such that

$$T_{(j_2,i)} = T_\beta^{r(i)} \circ \hat{T}_{(j_2,i)}$$

where  $\hat{T}_{(j_2,i)}$  is a (possibly trivial) Dehn multi-twist about the components of  $c_{j_2} - \tau_{j_2} - \beta$ . We note for later reference that property (b) was used here to ensure that the length of  $\beta_i$  at the endpoints of  $\zeta_i$  is at least  $\chi_1$ .

As  $\alpha, \beta \in c_{j_2} - \tau_{j_2}$  we have  $d_{\mathcal{C}}(\alpha, \beta) \leq 1$  and hence by invariance under the action of the mapping class group, for sufficiently large  $i$  the distance in  $\mathcal{CG}(S)$  between  $\alpha_i$  and

$$\beta_i = (T_{(j_2-1,i)} \circ \dots \circ T_{(1,i)})^{-1} \beta$$

is at most one. By property (b) above and the choice of  $p$ , this implies that the  $\zeta_i(0)$ -length of  $\beta_i$  and the  $\zeta_i(\sigma_i)$ -length of  $\beta_i$  is at least  $\chi_1$ . Moreover, the curve  $\beta_i$  becomes short along  $\zeta_i$ .

Using the notations in the previous paragraph, we consider now the case that  $\beta$  does not intersect any of the multi-curves  $c_j$  ( $0 \leq j \leq k$ ). Then  $\beta$  is invariant under each of the Dehn multi-twists  $T_{(j,i)}$ . Thus we have  $\beta_i = \beta$  for all  $i$  and hence  $\ell_\beta(\zeta_i(0)) \geq \chi_1, \ell_\beta(\zeta_i(\sigma_i)) \geq \chi_1$ . Since the arcs  $T_{(j,i)} \circ \dots \circ T_{(1,i)} \zeta_i|[t_j, t_{j+1}]$  converge as  $i \rightarrow \infty$  to the arc  $\xi_j$ , the  $\zeta_i(0)$ -length and the  $\zeta_i(\sigma_i)$ -length of  $\beta$  is bounded from above independent of  $i$ . Moreover, the curve  $\beta$  becomes short along the geodesics  $\zeta_i$ .

Call a pants decomposition  $P$  of  $S$  a Bers decomposition for some  $x \in \overline{\mathcal{T}(S)}$  if the  $x$ -length of any component of  $P$  is at most  $\chi_0$  (this means in particular that  $P$  contains all nodes of  $x$ ). Length functions on  $\mathcal{T}(S)$  extend continuously to functions on  $\overline{\mathcal{T}(S)}$  with values in  $[0, \infty]$ . Thus by the collar lemma, there is a neighborhood  $U$  of  $x_0$  in  $\overline{\mathcal{T}(S)}$  such that the number of pants decompositions which are Bers decompositions for points in  $U$  is finite. Namely, if the simple closed curve  $\omega$  is a node of  $x_0$  then there is a neighborhood  $V$  of  $x_0$  in  $\overline{\mathcal{T}(S)}$  such that every Bers decomposition for any  $x \in V$  contains  $\omega$ . Consequently, by passing to another subsequence we may assume that the Bers decompositions  $P_i$  for  $\zeta_i(0)$  coincide for all  $i$ . We denote this common pants decomposition by  $P$ .

For each  $i$  let as before  $P_{\sigma_i}$  be a Bers decomposition for  $\zeta_i(\sigma_i)$ . We claim that

$$\text{diam}(\pi_\beta(P) \cup \pi_\beta(P_{\sigma_i})) \rightarrow \infty (i \rightarrow \infty).$$

Namely, the Dehn twist  $T_\beta$  about  $\beta$  commutes with each of the Dehn multi-twists  $T_{(j,i)}$ . Write  $\hat{T}_{(j,i)} = T_{(j,i)}$  for  $j \neq j_2$  and let as before  $\hat{T}_{(j_2,i)}$  be the (possibly trivial) Dehn multi-twist about the components of  $c_{j_2} - \tau_{j_2} - \beta$  such that  $T_{(j_2,i)} =$

$T_\beta^{r(i)} \circ \hat{T}_{(j_2, i)}$  for some  $r(i) \in \mathbb{Z}$ . Since by the above assumption  $\beta$  is disjoint from each of the multicurves  $c_j$  ( $j \leq k-1$ ) we have

$$T_{(k-1, i)} \circ \cdots \circ T_{(1, i)} = T_\beta^{r(i)} \circ (\hat{T}_{(k-1, i)} \circ \cdots \circ \hat{T}_{(1, i)})$$

for all  $i$ .

Now if  $\omega \in \mathcal{C}(S)$  is any simple closed curve and if  $T$  is any Dehn multi-twist about a simple closed curve  $\kappa \in \mathcal{C}(S) - \{\beta\}$  which is disjoint from  $\beta$  then  $\pi_\beta(\omega) = \pi_\beta(T\omega)$ . As a consequence, for each  $i$  we have

$$(4) \quad \pi_\beta(P_{\sigma_i}) = \pi_\beta(\hat{T}_{(k-1, i)} \circ \cdots \circ \hat{T}_{(1, i)}(P_{\sigma_i})) \subset \mathcal{CG}(\beta).$$

Proposition 23 of [W03] together with  $\text{Mod}(S)$ -invariance of the Weil-Petersson metric shows that the Weil-Petersson distance between the points

$$(T_{(k-1, i)} \circ \cdots \circ T_{(1, i)})^{-1}(x_k) = (\hat{T}_{(k-1, i)} \circ \cdots \circ \hat{T}_{(1, i)})^{-1}(T_\beta^{-r(i)}(x_k))$$

and the points  $\zeta_i(\sigma_i)$  converges to zero as  $i \rightarrow \infty$ . Using once more invariance of the Weil-Petersson metric under the mapping class group, we conclude that

$$d_{WP}(\hat{T}_{(k-1, i)} \circ \cdots \circ \hat{T}_{(1, i)}(\zeta_i(\sigma_i)), T_\beta^{-r(i)}(x_k)) \rightarrow 0 \ (i \rightarrow \infty).$$

Now there are only finitely many Bers decompositions for all points in a small neighborhood of  $x_k$  and therefore by invariance under the action of the mapping class group and by (4) above, up to passing to a subsequence there is a Bers decomposition  $\hat{P}$  for  $x_k$  such that

$$\pi_\beta(P_{\sigma_i}) = \pi_\beta(T_\beta^{-r(i)}\hat{P})$$

for all sufficiently large  $i$ . By property (2) for  $\beta$ , the pants decomposition  $P_{\sigma_i}$  has an essential intersection with  $\beta$  and hence the same holds true for  $\hat{P}$ .

Since the diameter in  $\mathcal{CG}(\beta)$  of the projection  $\pi_\beta(P) \cup \pi_\beta(\hat{P})$  is finite and since  $|r(i)| \rightarrow \infty$  ( $i \rightarrow \infty$ ), this implies that the diameters of the projections  $\pi_\beta(P) \cup \pi_\beta(P_{\sigma_i})$  tend to infinity with  $i$ . But  $P_i = P$  for all  $i$  and  $d_{\mathcal{C}}(\beta, \alpha_i) \leq 1$  and hence this contradicts the assumption (c) above.

For the proof of the proposition we are now left with the case that there is some  $j \in \{0, \dots, k\}$  such that  $\beta$  intersects the multi-curve  $c_j$ . Let  $j_1 \in \{0, \dots, j_2 - 1\}$  be the maximal number  $j \leq j_2 - 1$  such that  $i(c_j, \beta) \neq 0$ . If there is no such  $j$  then write  $j_1 = -1$ . Similarly, let  $j_3 \in \{j_2 + 1, \dots, k\}$  be the minimal number  $j \geq j_2 + 1$  such that  $i(c_j, \beta) \neq 0$ . If there is no such number then write  $j_2 = k + 1$ . By our assumption, we either have  $j_1 \geq 0$  or  $j_3 \leq k$ . For  $j_1 < j < j_3$ , the curve  $\beta$  is invariant under the Dehn multi-twist  $T_{(j, i)}$ . Define

$$\rho_i = T_{(j_1, i)} \circ \cdots \circ T_{(1, i)}\zeta_i.$$

If  $j_1 > -1$  then we have  $\ell_\beta(\rho_i(t_{j_1})) \rightarrow \infty$  ( $i \rightarrow \infty$ ),  $\ell_\beta(\rho_i(t_{j_1+1})) \rightarrow 0$  ( $i \rightarrow \infty$ ) and therefore by convexity of length functions along Weil-Petersson geodesics, for sufficiently large  $i$  there is a unique number  $s_i \in (t_{j_1}, t_{j_2})$  such that  $\ell_\beta(\rho_i(s_i)) = \chi_1$ . If  $j_1 = -1$  then using once more convexity and the assumption that the  $\zeta_i(0)$ -length of  $\beta_i$  is not smaller than  $\chi_1$  we also find a unique number  $s_i \in (t_{j_1}, t_{j_2})$  such that  $\ell_\beta(\rho_i(s_i)) = \chi_1$ . Similarly we find for all sufficiently large  $i$  a unique number  $u_i \in (t_{j_2}, t_{j_3})$  such that  $\ell_\beta(\rho_i(u_i)) = \chi_1$ . By passing to a subsequence, we

may assume that  $s_i \rightarrow s, u_i \rightarrow u$ . Using again Wolpert's gradient bound for length functions, we have  $s \in (t_{j_1}, t_{j_2}), u \in (t_{j_2}, t_{j_3})$  and  $s < u$ .

The Weil-Petersson geodesics  $\rho_i[s_i, u_i]$ , the Bers decompositions  $Q_i^0$  for  $\rho_i(s_i)$ ,  $Q_i^1$  for  $\rho_i(u_i)$  and the curve  $\beta$  have all properties required to apply the special case of the proposition established in the first part of this proof. The first part of this proof implies that the diameters of the projections

$$\pi_\beta(Q_i^0) \cup \pi_\beta(Q_i^1)$$

tend to infinity with  $i$ . By equivariance under the mapping class group, if

$$\beta_i = (T_{(j_1, i)} \circ \cdots \circ T_{(1, i)})^{-1} \beta$$

and if  $\hat{Q}_i^0, \hat{Q}_i^1$  are Bers decompositions for  $\zeta_i(s_i), \zeta_i(u_i)$ , then

$$(5) \quad \text{diam}(\pi_{\beta_i}(\hat{Q}_i^0) \cup \pi_{\beta_i}(\hat{Q}_i^1)) \rightarrow \infty$$

as well.

By equivariance of length functions under the action of  $\text{Mod}(S)$ , we have

$$\ell_\beta(\zeta_i(s_i)) = \chi_1, \ell_{\beta_i}(\zeta_i(u_i)) = \chi_1,$$

moreover the length of  $\beta_i$  become arbitrarily small along  $\zeta_i[s_i, u_i]$  as  $i \rightarrow \infty$ . Thus by convexity, the length of  $\beta_i$  is at least  $\chi_1$  on  $[0, s_i] \cup [u_i, \sigma_i]$ . Corollary 4.2, applied to the Weil-Petersson geodesics  $\zeta_i[0, s_i]$  and  $\zeta_i[u_i, \sigma_i]$  of length at most  $R$  and the curves  $\beta_i \in \mathcal{C}(S)$ , yields that

$$\max\{\text{diam}(\pi_{\beta_i}(P_i) \cup \pi_{\beta_i}(\hat{Q}_i^0)), \text{diam}(\pi_{\beta_i}(\hat{Q}_i^1) \cup \pi_{\beta_i}(P_{\sigma_i}))\} \leq 4\chi_1 R + 8\chi_1.$$

Together with (5) this implies that  $\text{diam}(\pi_{\beta_i}(P_i) \cup \pi_{\beta_i}(P_{\sigma_i})) \rightarrow \infty$  ( $i \rightarrow \infty$ ) which contradicts assumption (c) above. The proposition is proven.  $\square$

## 5. LENGTH BOUNDS IN THE THICK PART OF TEICHMÜLLER SPACE

In this section we use the results from Section 4 to estimate the length of the intersection of a family of Weil-Petersson geodesics with the thick part of Teichmüller space. In particular, we show that for every  $\epsilon > 0$ , a Weil-Petersson geodesic  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  which connects two points on a Teichmüller geodesic  $\gamma \subset \mathcal{T}(S)_\epsilon$  spends a fixed percentage of time in the thick part of Teichmüller space. The main two ingredients for the proof of this fact are Proposition 4.3 and the following consequence of hyperbolicity of the curve graph (which holds true for every hyperbolic geodesic metric space).

**Lemma 5.1.** *For every  $\kappa > 1$  there is a number  $b(\kappa) > 0$  and for every  $n > 0$  there are numbers  $\tau = \tau(\kappa, n) > 0$ ,  $T = T(\kappa, n) > 0$  with the following properties. Let  $k > 0$ , let  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  be any geodesic and let  $\omega : [0, r] \rightarrow \mathcal{CG}(S)$  be a one-Lipschitz curve of length  $r \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(r) = \rho(k)$ . Then there is a set  $A \subset \{0, \dots, k-1\}$  of cardinality at least  $\tau k$  and for every  $i \in A$  there are numbers  $r_i \in [0, r], s_i \in [r_i, r_i + T]$  with  $r_{i+1} \geq s_i$  and there is a number  $j_i \geq i + n$  so that*

$$d_{\mathcal{C}}(\omega(r_i), \rho(i)) \leq b(\kappa), d_{\mathcal{C}}(\omega(s_i), \rho(j_i)) \leq b(\kappa) \quad (i = 1, 2).$$

*Proof.* Let  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  be any geodesic in the curve graph. Let  $\Pi : \mathcal{CG}(S) \rightarrow \rho[0, k]$  be a shortest distance projection (i.e.  $\Pi$  associates to a point  $x \in \mathcal{CG}(S)$  a point  $\Pi(x) \in \rho[0, k]$  of minimal distance; note that such a point need not be unique, and the map  $\Pi$  need not be continuous). By hyperbolicity of  $\mathcal{CG}(S)$ , there are numbers  $a \geq 1, b > 0$  only depending on the hyperbolicity constant such that the projection  $\Pi$  satisfies the following contraction properties (see Section 2 of [MM99] for these properties in the case of the curve graph).

- (1) If  $d_{\mathcal{C}}(x, y) \leq 1$  then  $d_{\mathcal{C}}(\Pi(x), \Pi(y)) \leq a$ .
- (2) If  $d_{\mathcal{C}}(x, \Pi(x)) \geq a$  and  $d_{\mathcal{C}}(x, y) \leq bd_{\mathcal{C}}(x, \Pi(x))$  then  $d_{\mathcal{C}}(\Pi(x), \Pi(y)) \leq a$ .

As a consequence, the following holds true. For every  $p > a$  and for every one-Lipschitz edge path  $\theta : [0, s] \rightarrow \mathcal{CG}(S)$  of length  $s > 0$  which does not intersect the  $p$ -neighborhood of  $\rho[0, k]$ , we have

$$(6) \quad \text{diam}(\Pi\theta[0, s]) \leq as/bp + a.$$

Namely, by property (2) above, the claim holds true if  $d_{\mathcal{C}}(\theta(0), \theta(\sigma)) < bp$  for all  $\sigma \in [0, s]$ . Otherwise let  $s_1 \geq bp$  be the smallest number such that  $d_{\mathcal{C}}(\theta(0), \theta(s_1)) = bp$ . Then  $d_{\mathcal{C}}(\Pi\theta(0), \Pi\theta(\sigma)) \leq a$  for every  $\sigma \in [0, s_1]$ . Moreover,  $d_{\mathcal{C}}(\theta(s_1), \Pi\theta(s_1)) \geq p$  and hence we can repeat this argument for the interval  $[s_1, s_2]$  where  $s_2 \geq s_1 + bp$  is the smallest number such that  $d_{\mathcal{C}}(\theta(s_1), \theta(s_2)) = bp$ . The estimate (6) now follows by induction.

Let  $\kappa > 1$  and let  $\omega : [0, r] \rightarrow \mathcal{CG}(S)$  be any one-Lipschitz edge path of length  $r \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(r) = \rho(k)$ . Then property (1) above implies that the image of  $\omega[0, r]$  under the projection  $\Pi$  is  $a$ -dense in  $\rho[0, k]$ . This means that for every  $t \in [0, k]$  there is some  $s \in [0, r]$  such that  $d_{\mathcal{C}}(\Pi(\omega(s)), \rho(t)) \leq a$ . Moreover, we have  $\Pi(\omega(0)) = \rho(0)$  and  $\Pi(\omega(r)) = \rho(k)$ .

Identify  $\rho[0, k]$  with the line segment  $[0, k]$ . Let  $\ell > 0$  be the largest integer so that  $5a\ell \leq k$ ; then  $k/5a - 1 \leq \ell \leq k/5a$ . Choose inductively a sequence  $0 = q_0 < \dots < q_{\ell} < r \leq \kappa k$  by requiring that  $q_i$  is the smallest number such that  $d_{\mathcal{C}}(\Pi(\omega(q_i)), \rho(5ai)) \leq a$ . Note that

$$\Pi(\omega(q_i)) + 3a \leq \Pi(\omega(q_{i+1})) \leq \Pi(\omega(q_i)) + 7a \text{ for all } i.$$

Define

$$A = \{i \leq \ell \mid q_{i+1} \leq q_i + 15\kappa a\}.$$

Since the length of  $\omega$  does not exceed  $\kappa k$  and  $\ell + 1 \geq k/5a$ , the cardinality of the subset  $A$  of  $\{0, \dots, \ell\}$  exceeds  $2(\ell + 1)/3 > k/8a$ .

Now if  $i \in A$  then the arc  $\omega[q_i, q_{i+1}]$  of length at most  $15\kappa a$  is mapped by  $\Pi$  to a subset of  $[0, k]$  of diameter at least  $3a$ . Thus by the estimate (6), if  $\chi > 0$  is such that  $15\kappa a^2/b\chi + a = 3a$  then there is some  $j(i) \in [q_i, q_{i+1}]$  such that the distance between  $\omega(j(i))$  and  $\rho[0, k]$  is at most  $\chi$ . Since  $q_{i+1} \leq q_i + 15\kappa a$ , since  $\omega$  is a one-Lipschitz-curve and since the projection  $\Pi$  is distance minimizing, the distance between  $\omega(q_i)$  and  $\Pi\omega(q_i)$  is bounded from above by  $\chi + 15\kappa a$ , i.e. by a universal constant only depending on  $\kappa$ .

Let  $n \in [1, \ell - 1]$  and let  $T_n : \mathbb{Z} \rightarrow \mathbb{Z}$  be the translation  $T_n(z) = z - n$ . Since the cardinality of  $A \subset \{0, \dots, \ell\}$  is not smaller than  $2(\ell + 1)/3$ , the cardinality of the

intersection  $A \cap T_n(A)$  is at least  $(\ell + 1 - n)/3$ . The cardinality of a maximal subset  $C \subset A \cap T_n(A)$  with the additional property that if  $c \in C$  then  $p \notin C$  for every  $p \in \{c+1, \dots, c+n-1\}$  is at least  $(\ell + 1 - n)/3n \geq (k/5a - n)/3n = k/15an - 1/3$ .

If  $c \in C$  then  $c \in A$ ,  $c+n \in A$  and  $c+j \notin C$  for  $1 \leq j \leq n-1$ . Recall that  $c \in A$  implies that the distance between  $\omega(q_c)$  and  $\rho[0, k]$  is uniformly bounded.

Define  $\tau = \tau(\kappa, n) = 1/40\kappa an$  and  $T = T(\kappa, n) = 30\kappa an$ . Since the cardinality of  $C$  is not smaller than  $k/15an - 1$  and the length of  $\omega$  does not exceed  $\kappa k$ , there is a subset  $\hat{C} \subset C$  of cardinality at least  $\tau k$  such that  $q_{c+n} - q_c \leq T$  for every  $c \in \hat{C}$ .

By construction, if  $c \in \hat{C}$  then

$$\max\{d_C(\omega(q_c), \rho(5ac)), d_C(\omega(q_{c+n}), \rho(5a(c+n)))\} \leq \chi + 15\kappa a$$

and  $d_C(\rho(5ac), \rho(5a(c+n))) = 5an > n$  (recall that  $a \geq 1$ ). This shows the lemma with  $b(\kappa) = \chi + 15\kappa a$  and with  $r_c = q_c$  and  $s_c = q_{c+n}$  for  $c \in \hat{C}$ .  $\square$

To use Lemma 5.1 to obtain a control on the intersection of a Weil-Petersson geodesic with the thick part of Teichmüller space we have to compare the distances  $d_{WP}$  and  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$ .

The following statement is well known. We include it as a lemma for easy reference.

**Lemma 5.2.** *For all  $x, y \in \mathcal{T}(S)$  the following holds true.*

- (1)  $d_{WP}(x, y) \leq \sqrt{2\pi(2g-2+m)}d_{\mathcal{T}}(x, y)$ .
- (2) *There is a number  $L > 1$  such that  $d_C(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y)) \leq Ld_{WP}(x, y) + L$ .*

*Proof.* The first part of the lemma is due to Linch [Li74].

To show the second part of the lemma, let as before  $\chi_0 > 0$  be a Bers constant for  $S$ . Let  $\Upsilon_{\mathcal{P}} : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$  be any map which associates to a hyperbolic metric  $x \in \mathcal{T}(S)$  a Bers decomposition for  $x$ . By a result of Brock [B03], the map  $\Upsilon_{\mathcal{P}}$  is a quasi-isometry with respect to the Weil-Petersson metric on  $\mathcal{T}(S)$  and the combinatorial metric  $d_{\mathcal{P}}$  on the pants graph  $\mathcal{PG}(S)$ : There is a number  $L_1 > 0$  such that

$$(7) \quad d_{WP}(x, y)/L_1 - L_1 \leq d_{\mathcal{P}}(\Upsilon_{\mathcal{P}}x, \Upsilon_{\mathcal{P}}y) \leq L_1 d_{WP}(x, y) + L_1 \text{ for all } x, y \in \mathcal{T}(S).$$

Let  $\Psi : \mathcal{P}(S) \rightarrow \mathcal{C}(S)$  be a map which associates to a pants decomposition  $P \in \mathcal{P}(S)$  one of its component curves. If  $P, P'$  are pants decompositions with  $d_{\mathcal{P}}(P, P') = 1$  then  $P'$  can be obtained from  $P$  by an elementary move and hence  $P \cap P' \neq \emptyset$  (recall that by assumption, the surface  $S$  is not a once punctured torus or a four-punctured sphere). This implies that the distance in  $\mathcal{CG}(S)$  between any component of  $P$  and any component of  $P'$  is at most 2. Since the pants graph  $\mathcal{PG}(S)$  is a geodesic metric graph, we have

$$d_C(\Psi P_1, \Psi P_2) \leq 2d_{\mathcal{P}}(P_1, P_2) \text{ for all } P_1, P_2 \in \mathcal{P}(S).$$

Now the map  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$  which associates to a point  $x \in \mathcal{T}(S)$  a simple closed curve of  $x$ -length at most  $\chi_0$  can be chosen to coincide with  $\Psi \circ \Upsilon_{\mathcal{P}}$ . Together with inequality (7), the second part of the lemma follows.  $\square$

As before, let  $\chi_0 > 0$  be a Bers constant for  $S$ . By Lemma 2.4, if  $\alpha, \beta \in \mathcal{C}(S)$  are such that there is some  $x \in \mathcal{T}(S)$  with  $\ell_{\alpha}(x) \leq \chi_0, \ell_{\beta}(x) \leq \chi_0$  the distance in  $\mathcal{CG}(S)$  between  $\alpha, \beta$  is at most  $a(\chi_0) > 0$ . In particular, if  $\alpha \in \mathcal{C}(S)$  is of  $x$ -length at most  $\chi_0$  and if  $\beta \in \mathcal{C}(S)$  is such that  $d_{\mathcal{C}}(\alpha, \beta) \geq a(\chi_0) + 3$  then  $\beta \cup \xi$  binds  $S$  for any simple closed curve  $\xi \in \mathcal{C}(S)$  with  $\ell_{\xi}(x) \leq \chi_0$ .

For  $\alpha \in \mathcal{C}(S)$  recall from Section 4 the definition of the graph  $\mathcal{CG}(\alpha)$  and the definition of the projection  $\pi_{\alpha} : \mathcal{C}(S) \rightarrow \mathcal{CG}(\alpha)$ . The following lemma is a version of a result of Masur and Minsky [MM00]: Teichmüller geodesics in the thick part of Teichmüller space have bounded combinatorics (see also [R05] and [R14]).

**Lemma 5.3.** *For every  $R > 0$  there are numbers  $\delta = \delta(R) > 0$ ,  $c = c(R) > 0$  with the property that for every Teichmüller geodesic arc  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  of length  $r \leq R$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(r)) \geq 2a(\chi_0) + 3$  the following is satisfied.*

- (1)  $\gamma[0, r] \subset \mathcal{T}(S)_{\delta}$ .
- (2)  $\text{diam}(\pi_{\beta}(\Upsilon_{\mathcal{T}}\gamma(0)) \cup \pi_{\beta}(\Upsilon_{\mathcal{T}}\gamma(r))) \leq c$  for all  $\beta \in \mathcal{C}(S)$ .

*Proof.* For  $R > 0$  a Teichmüller geodesic  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  of length  $r \leq R$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(r)) \geq 2a(\chi_0) + 3$  is entirely contained in  $\mathcal{T}(S)_{\delta}$  where  $\delta = \chi_0 e^{-R}$ . Namely, by Lemma 3.1 of [W79], if  $\zeta : J \subset \mathbb{R} \rightarrow \mathcal{T}(S)$  is any Teichmüller geodesic and if  $s \in J$ ,  $\alpha \in \mathcal{C}(S)$  are such that  $\ell_{\alpha}(\zeta(s)) \leq \delta$  then  $\ell_{\alpha}(\zeta(t)) \leq \chi_0$  for every  $t$  with  $|s - t| \leq \log \chi_0 - \log \delta$ . In particular, by Lemma 2.4, if  $|s - t_i| \leq \log \chi_0 - \log \delta$  for  $i = 1, 2$  then we have  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\zeta(t_1)), \Upsilon_{\mathcal{T}}(\zeta(t_2))) \leq 2a(\chi_0)$ . The first part of the lemma follows.

The second part of the lemma can be extracted from [MM00]. Perhaps the most convenient reference is the distance formula of [R07b].  $\square$

We use Proposition 4.3 and Lemma 5.1-5.3 to show that a WP-geodesic connecting two sufficiently far apart points on a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  whose image under  $\Upsilon_{\mathcal{T}}$  makes controlled progress in the curve graph spends a definitive proportion of time in the thick part of Teichmüller space.

**Proposition 5.4.** *For every  $\theta > 0$  there are numbers  $\delta = \delta(\theta) > 0$  and  $\eta = \eta(\theta) > 0$  with the following property. Let  $\xi \geq 1/\eta$  and let  $\gamma : [0, \xi] \rightarrow \mathcal{T}(S)$  be a Teichmüller geodesic such that*

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(0)), \Upsilon_{\mathcal{T}}(\gamma(\xi))) \geq \theta\xi.$$

*If  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  is the Weil-Petersson geodesic connecting  $\zeta(0) = \gamma(0)$  to  $\zeta(\sigma) = \gamma(\xi)$  then  $\ell_{\delta\text{-thick}}(\zeta) \geq \eta\xi$ .*

*Proof.* Let  $\theta > 0$  and let  $\gamma : [0, \xi] \rightarrow \mathcal{T}(S)$  be any Teichmüller geodesic of length  $\xi \geq 2a(\chi_0) + 3/\theta$  such that

$$(8) \quad d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(0)), \Upsilon_{\mathcal{T}}(\gamma(\xi))) \geq \theta\xi.$$

Here  $a(\chi_0) > 0$  is as in Lemma 2.4 for a Bers constant  $\chi_0$  for  $S$ .

Write  $C = \sqrt{2\pi(2g - 2 + m)}$ . Lemma 5.2 yields that

$$(9) \quad \begin{aligned} d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(\xi))/L - 1 &\leq d_{WP}(\gamma(0), \gamma(\xi)) \\ &\leq C\xi = Cd_{\mathcal{T}}(\gamma(0), \gamma(\xi)) \leq Cd_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(\xi))/\theta. \end{aligned}$$

Let  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  be the WP-geodesic connecting  $\zeta(0) = \gamma(0)$  to  $\zeta(\sigma) = \gamma(\xi)$ . By the second part of Lemma 5.2, we have  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\zeta(i)), \Upsilon_{\mathcal{T}}(\zeta(i+1))) \leq 2L$  for all  $i$ . Thus via connecting  $\Upsilon_{\mathcal{T}}(\zeta(i))$  to  $\Upsilon_{\mathcal{T}}(\zeta(i+1))$  by a geodesic segment in  $\mathcal{CG}(S)$  for each  $i$  and subsequent reparametrization we obtain a 1-Lipschitz curve  $\omega : [0, u] \rightarrow \mathcal{CG}(S)$  connecting  $\omega(0) = \Upsilon_{\mathcal{T}}(\gamma(0))$  to  $\omega(u) = \Upsilon_{\mathcal{T}}(\gamma(\xi))$  whose length  $u$  does not exceed  $2Ld_{WP}(\gamma(0), \gamma(\xi))$ . (To obtain this estimate we slightly adjust the constant  $L$  to accommodate for the problem that the length  $\sigma$  of  $\zeta$  is not integral in general. This adjustment simplifies the notation).

Inequality (9) then implies that the length  $u$  of  $\omega$  is also bounded from above by

$$u \leq 2LCd_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(\xi))/\theta = \kappa d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(\xi))$$

where  $\kappa = 2LC/\theta > 0$  only depends on  $\theta$ . Moreover, every point on the curve  $\omega$  is of distance at most  $L$  from a point in  $\Upsilon_{\mathcal{T}}(\zeta[0, \sigma])$ .

Let  $k = d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(\xi)) \geq \max\{\theta\xi, u/\kappa\}$ . By Theorem 2.6, there is a number  $L_1 > 1$  such that the curve  $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$  is an unparametrized  $L_1$ -quasi-geodesic in the curve graph. Hence by hyperbolicity, the Hausdorff distance between its image and the image of a geodesic  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  with the same endpoints is bounded from above by a universal constant  $\beta > 0$ .

Let  $p = p(\chi_0, \chi_1) \geq 1$  be as defined before Proposition 4.3. For  $\kappa = 2LC/\theta > 1$  let  $b(\kappa) > 0$  be as in Lemma 5.1 and write

$$B = b(\kappa) + \beta + L.$$

Note that  $B > 0$  only depends on  $\theta$ .

Apply Lemma 5.1 to the geodesic  $\rho$  in  $\mathcal{CG}(S)$  of length  $k$ , the one-Lipschitz edge path  $\omega : [0, u] \rightarrow \mathcal{CG}(S)$  of length  $u \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(u) = \rho(k)$  and the constant

$$n = 2p + 4B + 2\beta + 5a(\chi_0) + 8.$$

Note that  $n > 0$  only depend on  $\theta$ . Using the fact that every point on  $\omega[0, u]$  is of distance at most  $L$  from a point in  $\Upsilon_{\mathcal{T}}(\zeta[0, \sigma])$  and that the Hausdorff distance between  $\Upsilon_{\mathcal{T}}\gamma[0, \xi]$  and  $\rho[0, k]$  is at most  $\beta$ , we conclude that there are numbers  $\tau = \tau(\theta) > 0$  and  $T = T(\theta) > 0$  and there is a subset  $A_0$  of  $\{1, \dots, k\}$  of cardinality at least  $\tau k$  and for each  $i \in A_0$  there is some  $r_i \in [0, \sigma]$  and some  $t_i \in [0, \xi]$  such that the following holds true.

- (1)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\zeta(r_i)), \Upsilon_{\mathcal{T}}(\gamma(t_i))) \leq B$ .
- (2) There are numbers  $e_i \leq T, v_i > 0$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(r_i + e_i), \Upsilon_{\mathcal{T}}\gamma(t_i + v_i)) \leq B$ .
- (3)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(t_i), \Upsilon_{\mathcal{T}}\gamma(t_i + v_i)) \geq 2p + 4B + 5a(\chi_0) + 8$ .
- (4)  $r_{i+1} \geq r_i + e_i, t_{i+1} \geq t_i + v_i$ .



Since the length  $\xi$  of the Teichmüller geodesic arc  $\gamma$  does not exceed  $k/\theta$  and since the cardinality of  $A_0$  is at least  $\tau k$ , there is a subset  $A_1$  of  $A_0$  of cardinality at least  $\tau k/2$  such that  $v_i \leq 2/\theta\tau$  for every  $i \in A_1$ . Lemma 5.3 then yields the existence of a number  $c > 0$  only depending on  $\theta$  such that for every  $i \in A_1$  and for every  $\alpha \in \mathcal{C}(S)$  we have

$$(10) \quad \text{diam}(\pi_\alpha(\Upsilon_{\mathcal{T}}\gamma(t_i)), \pi_\alpha(\Upsilon_{\mathcal{T}}\gamma(t_i + v_i))) \leq c.$$

Let  $i \in A_1$  and let  $\varphi, \psi \in \mathcal{C}(S)$  be such that

$$d_{\mathcal{C}}(\varphi, \Upsilon_{\mathcal{T}}\gamma(t_i)) \leq B, d_{\mathcal{C}}(\psi, \Upsilon_{\mathcal{T}}\gamma(t_i + v_i)) \leq B.$$

If  $\alpha \in \mathcal{C}(S)$  is such that  $d_{\mathcal{C}}(\varphi, \alpha) \geq B + 3, d_{\mathcal{C}}(\psi, \alpha) \geq B + 3$  then a geodesic in  $\mathcal{CG}(S)$  connecting  $\varphi, \psi$  to  $\Upsilon_{\mathcal{T}}(\gamma(t_i)), \Upsilon_{\mathcal{T}}(\gamma(t_i + v_i))$  does not intersect the ball of radius 2 about  $\alpha$ . In particular, each of the vertices of  $\mathcal{CG}(S)$  passed through by these geodesics intersects  $\alpha$  transversely. Now by Lemma 2.3 of [MM00], if  $\beta_1, \beta_2 \in \mathcal{C}(S)$  are two curves of distance one in  $\mathcal{CG}(S)$  which intersect  $\alpha$  transversely then the diameter in  $\mathcal{CG}(\alpha)$  of the projection  $\pi_\alpha(\beta_1) \cup \pi_\alpha(\beta_2)$  is at most 2. An inductive application of this fact to a geodesic in  $\mathcal{CG}(S)$  connecting  $\varphi, \psi$  to  $\Upsilon_{\mathcal{T}}(\gamma(t_i)), \Upsilon_{\mathcal{T}}(\gamma(t_i + v_i))$  shows that

$$\text{diam}(\pi_\alpha(\varphi) \cup \pi_\alpha(\Upsilon_{\mathcal{T}}\gamma(t_i))) \leq B + 1, \text{diam}(\pi_\alpha(\psi) \cup \pi_\alpha(\Upsilon_{\mathcal{T}}\gamma(t_i + v_i))) \leq B + 1.$$

Together with the estimate (10) we obtain that

$$(11) \quad \text{diam}(\pi_\alpha(\varphi) \cup \pi_\alpha(\psi)) \leq c + 2B + 2.$$

For  $i \in A_1$  (where  $A_1 \subset \{1, \dots, k\}$  is as above) consider the WP-geodesic arc  $\zeta[s_i, s_i + e_i]$ . By the properties (1)-(3) above, we have

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(s_i), \Upsilon_{\mathcal{T}}\zeta(s_i + e_i)) \geq 2p + 2B + 5a(\chi_0) + 8.$$

Even though the map  $t \rightarrow \Upsilon_{\mathcal{T}}(\zeta(t))$  is not continuous, by the choice of the constant  $a(\chi_0)$  the set  $\Upsilon_{\mathcal{T}}\zeta[0, \sigma]$  is  $a(\chi_0)$ -dense in a simplicial arc connecting  $\Upsilon_{\mathcal{T}}(\zeta(0))$  to  $\Upsilon_{\mathcal{T}}(\zeta(\sigma))$ . Therefore there is a number  $t \in (s_i, s_i + e_i)$  so that

$$(12) \quad \begin{aligned} d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(t), \Upsilon_{\mathcal{T}}\zeta(s_i)) &\geq p + B + 2a(\chi_0) + 4, \\ d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(t), \Upsilon_{\mathcal{T}}\zeta(s_i + e_i)) &\geq p + B + 2a(\chi_0) + 4. \end{aligned}$$

Note that by the second part of Lemma 5.2, there is a number  $v_0 > 0$  such that  $\min\{|s_i - t|, |s_i + e_i - t|\} \geq v_0$ .

If  $P_i, Q_i$  are Bers decompositions for  $\zeta(s_i), \zeta(s_i + e_i)$ , then using once more the definition of  $a(\chi_0)$ , inequality (12) implies that the distance in  $\mathcal{C}(S)$  between any component of  $P_i, Q_i$  and every component  $\psi$  of a Bers decomposition for  $\zeta(t)$  is at least  $p + B + 4$ . Thus the estimate (11), applied to components  $\varphi, \psi$  of  $P_i, Q_i$  and to every  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\xi, \beta) \leq 1$  (which is possible by the property (1) above) yields that

$$\text{diam}(\pi_\beta(P_i) \cup \pi_\beta(Q_i)) \leq c + 2B + 2.$$

Now  $\psi$  is an arbitrary component of a Bers decomposition for  $\zeta(t)$ , and the length  $e_i$  of the arc  $\zeta_i[s_i, s_i + e_i]$  does not exceed  $T$ . Therefore Proposition 4.3 shows the existence of a number  $\delta = \delta(\theta) > 0$  only depending on  $\theta$  such that  $\zeta(t) \in \mathcal{T}(S)_{2\delta}$ . Since by Wolpert's results the Weil-Petersson distance between  $\mathcal{T}(S)_{2\delta}$  and  $\mathcal{T}(S) - \mathcal{T}(S)_\delta$  is bounded from below by a universal constant (compare

the beginning of the proof of Lemma 4.1 for a more precise statement), there is a number  $v \leq v_0$  such that  $\ell_{\delta\text{-thick}}(\zeta[s_i, s_i + e_i]) \geq v$ .

Recall that  $i \in A_1$  was arbitrary and that the cardinality of  $A_1$  is not smaller than  $\tau k/2$ . Then the above consideration shows that  $\ell_{\delta\text{-thick}}(\zeta) \geq \tau k v/2 \geq \tau \xi v/2\theta$ . Since  $\tau, v$  only depend on  $\theta$ , the proposition follows.  $\square$

## 6. CONJUGATING THE FLOWS

For a  $\Phi_{\mathcal{T}}^t$ -invariant Borel subset  $A$  of  $\mathcal{Q}(S)$  define a *measurable conjugacy* of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $A$  into the geodesic flow of the Weil-Petersson metric to be an injective measurable map  $\Lambda : A \rightarrow \mathcal{Q}_{WP}(S)$  such that there is a measurable function  $\psi : A \times \mathbb{R} \rightarrow \mathbb{R}$  with the following properties.

- (1)  $\psi(x, 0) = 0$  for all  $x \in A$ .
- (2) For each fixed  $x \in A$  the function  $\psi(x, \cdot) : s \rightarrow \psi(x, s)$  is an increasing homeomorphism.
- (3)  $\Lambda(\Phi_{\mathcal{T}}^t x) = \Phi_{WP}^{\psi(x, t)} \Lambda(x)$  for all  $x \in A, t \in \mathbb{R}$ .

The goal of this section is to construct such a measurable conjugacy  $\Lambda$  on a  $\Phi_{\mathcal{T}}^t$ -invariant Borel subset  $\mathcal{E}$  of  $\mathcal{Q}(S)$  which has full measure for every  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure. The restriction of the map to any compact invariant subset of  $\mathcal{Q}(S)$  is continuous. We use this to establish the first part of Theorem 3.

We begin with isolating properties of typical orbits for  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures. We continue to use the assumptions and notations from Sections 2-6. Let again  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$  be a map which associates to a point  $x \in \mathcal{T}(S)$  a Bers curve on  $x$ , i.e. a simple closed curve of  $x$ -length at most  $\chi_0$  where  $\chi_0 > 0$  is a Bers constant for  $S$ . Our first goal is to study the distances  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(t))$  as  $t \rightarrow \infty$  for a Teichmüller geodesic  $\gamma$  whose cotangent line projects to an orbit of the flow  $\Phi_{\mathcal{T}}^t$  on  $\mathcal{Q}(S)$  which is typical for an invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ .

For this we face the problem that the map  $\Upsilon_{\mathcal{T}}$  depends on choices and may not be measurable, and the same problem may arise for the function  $(x, y) \rightarrow d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y))$  on  $\mathcal{T}(S) \times \mathcal{T}(S)$ . We resolve this problem by using a construction from [H10a].

Namely, choose a smooth function  $\sigma : [0, \infty) \rightarrow [0, 1]$  with  $\sigma[0, \chi_0] \equiv 1$  and  $\sigma[2\chi_0, \infty) \equiv 0$ . For every  $h \in \mathcal{T}(S)$  we obtain a finite Borel measure  $\mu_h$  on  $\mathcal{C}(S)$  by defining

$$\mu_h = \sum_{\beta} \sigma(\ell_h(\beta)) \delta_{\beta}$$

where  $\delta_{\beta}$  denotes the Dirac mass at  $\beta$ . By the collar lemma, the number of simple closed geodesics on  $(S, h)$  of length at most  $2\chi_0$  is bounded from above independent of  $h$ . Thus the total mass of  $\mu_h$  is bounded from above and below by a universal positive constant, and by Lemma 2.4, the diameter in  $\mathcal{CG}(S)$  of the support of  $\mu_h$  is uniformly bounded as well.

Define a symmetric non-negative function  $\rho$  on  $\mathcal{T}(S) \times \mathcal{T}(S)$  by

$$\rho(h, h') = \int_{\mathcal{C}(S) \times \mathcal{C}(S)} d_{\mathcal{C}}(\cdot, \cdot) d\mu_h \times d\mu_{h'} / \mu_h(\mathcal{C}(S)) \mu_{h'}(\mathcal{C}(S)).$$

Lemma 3.3 of [H10a] shows that the function  $\rho$  is continuous and invariant under the diagonal action of  $\text{Mod}(S)$ . Moreover, there is a universal constant  $a_0 > 0$  such that

$$(13) \quad \rho(h, h') - a_0 \leq d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(h), \Upsilon_{\mathcal{T}}(h')) \leq \rho(h, h') + a_0 \text{ for all } h, h' \in \mathcal{T}(S).$$

For  $q \in \mathcal{Q}(S)$  and for  $t \geq 0$  we can now define a number  $r(q, t) \geq 0$  as follows. A lift  $\tilde{q}$  of  $q$  to  $\tilde{\mathcal{Q}}(S)$  determines a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  with initial unit cotangent  $\tilde{q}$ . By invariance of the function  $\rho$  under the diagonal action of the mapping class group, the number

$$r(q, t) = \rho(\gamma(0), \gamma(t))$$

does not depend on the choice of  $\tilde{q}$  and hence it defines a continuous function  $r : \mathcal{Q}(S) \times \mathbb{R} \rightarrow [0, \infty)$ . We have.

**Lemma 6.1.** *Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . Then there is a number  $b = b(\mu) > 0$  such that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} r(q, t) = b$$

for  $\mu$ -almost every  $q \in \mathcal{Q}(S)$ .

*Proof.* By the triangle inequality for  $d_{\mathcal{C}}$  and the estimate (13) above, for  $q \in \mathcal{Q}(S)$  and  $s, t > 0$  we have

$$r(q, s+t) \leq r(q, s) + r(\Phi_{\mathcal{T}}^s q, t) + 2a_0$$

where  $a_0 > 0$  is as in (13). Thus by the subadditive ergodic theorem [Kr85], for  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} r(q, t)$$

exists and does not depend on  $q$ . (Here we apply the subadditive ergodic theorem to the function  $(q, t) \rightarrow r(q, t) + 2a_0$ ).

We have to show that this limit is positive almost everywhere. For this recall from Theorem 2.6 that there is a number  $L_1 > 0$  such that the image under  $\Upsilon_{\mathcal{T}}$  of any Teichmüller geodesic is an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ . By the Poincaré recurrence theorem, the  $\Phi_{\mathcal{T}}^t$ -orbit of  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  is recurrent under the Teichmüller flow. Therefore the support of the vertical measured geodesic lamination of  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  fills  $S$  and is uniquely ergodic [M82]. As a consequence, if  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  is a geodesic whose unit cotangent line is a lift of an orbit of  $\Phi_{\mathcal{T}}^t$  which is typical for  $\mu$  then the map  $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$  is an unparametrized  $L_1$ -quasi-geodesic of infinite length [Kl99, H06]. This means that  $r(q, t) \rightarrow \infty$  ( $t \rightarrow \infty$ ) for  $\mu$ -almost every  $q$ . Moreover, by Lemma 2.4 of [H10a] and the estimate (13) above, there is a number  $c > 0$  such that

$$(14) \quad r(q, t+s) - c \leq r(q, s) + r(\Phi_{\mathcal{T}}^s q, t) \leq r(q, t+s) + c \text{ for all } q \in \mathcal{Q}(S), s, t > 0.$$

Now the function  $r : \mathcal{Q}(S) \times \mathbb{R} \rightarrow [0, \infty)$  is continuous and the measure  $\mu$  is Borel regular. Hence there is a compact subset  $A$  of  $\mathcal{Q}(S)$  with  $\mu(A) > 3/4$  and a number  $t_0 > 0$  such that  $r(q, t_0) \geq 4c$  for all  $q \in A$  where  $c > 0$  is as in (14) above. For the continuous non-negative function  $\varphi(q) = r(q, t_0)$  on  $\mathcal{Q}(S)$  we then have  $\int \varphi d\mu \geq 3c$ . The Birkhoff ergodic theorem implies that

$$\frac{1}{t_0} \int_0^{t_0} \left( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\Phi_{\mathcal{T}}^{it_0} \Phi_{\mathcal{T}}^s q) \right) ds \geq 3c$$

for  $\mu$ -almost every  $q$ . The estimate (14) shows that  $r(q, nt_0) \geq \sum_{i=0}^{n-1} \varphi(\Phi_{\mathcal{T}}^{it_0} q) - nc$  and therefore

$$\liminf_{n \rightarrow \infty} \frac{1}{nt_0} r(q, nt_0) \geq 2c$$

for  $\mu$ -almost every  $q$ . This completes the proof of the lemma.  $\square$

For  $j > 0, \ell > 0$  define

$$B^+(j, \ell) = \{q \in \mathcal{Q}(S) \mid r(q, t) \geq t/j \text{ for all } t \geq \ell\}.$$

**Lemma 6.2.** *The set  $B^+(j, \ell)$  is compact and consists of quadratic differentials with uniquely ergodic vertical measured lamination.*

*Proof.* Since the function  $r$  is continuous, we have

$$B^+(j, \ell) = \bigcap_{s \geq \ell, s \in \mathbb{Q}} \{q \mid r(q, s) \geq s/j\}$$

and hence  $B^+(j, \ell)$  is a countable intersection of closed sets. In particular,  $B(j, \ell) \subset \mathcal{Q}(S)$  is closed.

For  $\epsilon > 0$  let  $\mathcal{M}(S)_\epsilon$  be the projection of  $\mathcal{T}(S)_\epsilon$  into the moduli space of curves. Let  $q$  be a quadratic differential whose vertical measured geodesic lamination is not uniquely ergodic. Then for every  $\epsilon > 0$  there is some  $\tau(\epsilon) > 0$  such that for every  $s \geq \tau(\epsilon)$ , the surface underlying  $\Phi_{\mathcal{T}}^s(q)$  is not contained in  $\mathcal{M}(S)_\epsilon$  [M82]. By Lemma 3.1 of [W79], if  $\alpha$  is a curve of length at most  $\epsilon$  on the surface underlying  $\Phi_{\mathcal{T}}^s(q)$ , then the length of  $\alpha$  on the surface underlying  $\Phi_{\mathcal{T}}^u(q)$  is at most  $\chi_0$  for every  $u \in [s - \log(\chi_0 - \epsilon), s + \log(\chi_0 - \epsilon)]$ . This implies that if  $\epsilon$  is sufficiently small then we have  $r(z, s) \leq s/2j$  for all sufficiently large  $s$  and every quadratic differential  $z$  with the property that the projection of the forward orbit  $\{\Phi_{\mathcal{T}}^t z \mid t \geq 0\}$  of  $z$  does not intersect  $\mathcal{M}(S)_\epsilon$ .

By the definition of the set  $B^+(j, \ell)$ , this shows first that  $B(j, \ell)$  projects into  $\mathcal{M}(S)_\epsilon$  for a number  $\epsilon > 0$  depending on  $j, \ell$ . In particular,  $B(j, \ell)$  is compact. Moreover, by the estimate (14), the orbit of  $q \in B^+(j, \ell)$  under  $\Phi_{\mathcal{T}}^t$  recurs to a fixed compact set for arbitrarily large times hence its vertical measured lamination is uniquely ergodic [M82]:  $\square$

Let  $\mathcal{F} : \mathcal{Q}(S) \rightarrow \mathcal{Q}(S)$  be the *flip*  $\mathcal{F}(q) = -q$ . Define

$$B(j, \ell) = B^+(j, \ell) \cap \mathcal{F}(B^+(j, \ell)),$$

$$E^+(j, \ell) = B^+(j, \ell) \cap \limsup_{i \rightarrow \infty} \Phi^{-i} B^+(j, \ell) = B(j, \ell) \cap (\bigcap_{u=1}^{\infty} (\bigcup_{i=u}^{\infty} \Phi^{-i} B(j, \ell))),$$

$$E(j, \ell) = E^+(j, \ell) \cap \mathcal{F}(E^+(j, \ell)) \text{ and } \mathcal{E} = \bigcup_j (\bigcup_{\ell} E(j, \ell)).$$

- Lemma 6.3.** (1)  $\mathcal{E}$  is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel subset of  $\mathcal{Q}(S)$ .  
 (2)  $\mu(\mathcal{E}) = 1$  for every  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure.  
 (3) The horizontal and vertical measured geodesic laminations of every  $q \in \mathcal{E}$  are uniquely ergodic.  
 (4) Every compact  $\Phi_{\mathcal{T}}^t$ -invariant subset of  $\mathcal{Q}(S)$  is contained in  $\mathcal{E}$ .

*Proof.* That  $\mathcal{E}$  is a Borel set is immediate from Lemma 6.2 and the definitions. Moreover, Lemma 6.2 shows that the horizontal and vertical measured geodesic laminations of every  $q \in \mathcal{E}$  are uniquely ergodic.

Part (2) of the lemma follows from Lemma 6.1 and the fact that the image under the flip of any  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure on  $\mathcal{Q}(S)$  is an invariant Borel probability measure.

To show invariance of  $\mathcal{E}$  under  $\Phi_{\mathcal{T}}^t$ , it suffices to show that  $\cup_j(\cup_{\ell} E^+(j, \ell))$  is invariant. For this recall that  $E^+(j, \ell)$  is the set of all points  $q \in B^+(j, \ell)$  so that  $\Phi_{\mathcal{T}}^t q \in B^+(j, \ell)$  for infinitely many  $i > 0$ . Thus if  $q \in E^+(j, \ell)$  and if  $t > 0$  then there is some  $u \geq 0$  with  $\Phi_{\mathcal{T}}^u(\Phi_{\mathcal{T}}^t q) \in E^+(j, \ell)$ . Inequality (14) shows that

$$r(\Phi_{\mathcal{T}}^t(q), s) \geq r(\Phi_{\mathcal{T}}^{u+t}(q), s - u) - c \geq (s - u)/j - c$$

for all  $s \geq \ell + u$  and hence putting  $\ell' = 2 \max\{\ell + u, 2c\}$  we conclude that  $\Phi_{\mathcal{T}}^t(q) \in B(2j, \ell')$ . The same argument also shows that  $\Phi_{\mathcal{T}}^t q \in E^+(2j, \ell')$ .

Part (4) of the lemma is an immediate consequence of the fact that the image under  $\Upsilon$  of a Teichmüller geodesic which is entirely contained in  $\mathcal{T}(S)_{\epsilon}$  for some  $\epsilon > 0$  is a parametrized quasi-geodesic in  $\mathcal{CG}(S)$ .  $\square$

The idea for constructing the conjugacy is to find for a Teichmüller geodesic with uniquely ergodic horizontal and vertical measured laminations a Weil-Petersson geodesic which has these laminations as ending measures. For this we need a technical preparation which we formulate as two lemmas.

**Lemma 6.4.** *Let  $\gamma_i : \mathbb{R} \rightarrow \mathcal{T}(S)$  be a sequence of Teichmüller geodesics converging locally uniformly to a Teichmüller geodesic  $\gamma$ . Assume that the vertical measured geodesic lamination  $\nu$  of the quadratic differential which defines  $\gamma$  is uniquely ergodic. Let  $n_i \rightarrow \infty$  and for each  $i$  let  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$  be the WP-geodesic connecting  $\zeta_i(0) = \gamma_i(0)$  to  $\zeta_i(\sigma_i) = \gamma_i(n_i)$ . Then up to passing to a subsequence, the geodesics  $\zeta_i$  converge locally uniformly to an infinite WP-ray  $\zeta : [0, \infty) \rightarrow \mathcal{T}(S)$ . The length of  $\nu$  is bounded along  $\zeta$ .*

*Proof.* By Theorem 2.6, there is a number  $L_1 > 0$  such that the image under  $\Upsilon_{\mathcal{T}}$  of every Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  is an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ . By hyperbolicity of the curve graph, this means in particular that there is a number  $b > 0$  and there is a geodesic  $\rho : [0, b) \rightarrow \mathcal{CG}(S)$  such that the Hausdorff distance between  $\rho(J)$  and  $\Upsilon_{\mathcal{T}}(\gamma[0, \infty))$  is bounded from above by a universal constant  $p_1 > 0$ .

Let  $\gamma_i, \gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be as in the lemma. Let  $\nu$  be the vertical measured geodesic lamination of  $\gamma$ , normalized in such a way that the  $\gamma(0)$ -length of  $\nu$  equals one. By assumption,  $\nu$  is uniquely ergodic. This implies that the diameter of  $\Upsilon_{\mathcal{T}}(\gamma[0, \infty))$

is infinite and the support of  $\nu$  defines a point in the boundary  $\partial\mathcal{CG}(S)$  of the curve graph  $\mathcal{CG}(S)$ . If  $(\beta_i) \subset \mathcal{C}(S)$  is any sequence converging in  $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$  to the support of  $\nu$  then by unique ergodicity of  $\nu$ , the projective measured geodesic laminations  $[\beta_i]$  supported in  $\beta_i$  converge in  $\mathcal{PML}$  to the projective measured geodesic lamination  $[\nu]$  which is the class of  $\nu$  (see Theorem 1.4 of [K199] and Theorem 1.1 of [H06]).

Since  $\gamma_i \rightarrow \gamma$  uniformly on compact sets, as  $i \rightarrow \infty$  longer and longer subarcs of the uniform unparametrized quasi-geodesic  $\Upsilon_{\mathcal{T}} \circ \gamma$  are uniformly fellow-traveled by subarcs of the uniform unparametrized quasi-geodesics  $\Upsilon_{\mathcal{T}} \circ \gamma_i$ . Thus for any sequence  $n_i \rightarrow \infty$ , the curves  $c_i = \Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  converge in  $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$  to the support of  $\nu$  (compare the more detailed argument in the proof of Proposition 3.4 of [H10a]). As a consequence, we have  $[c_i] \rightarrow [\nu]$  in  $\mathcal{PML}$ .

For  $i > 0$  let  $\zeta_i$  be the WP-geodesic connecting  $\zeta_i(0) = \gamma_i(0)$  to  $\zeta_i(\sigma_i) = \gamma(n_i)$ . After passing to a subsequence we may assume that the directions  $v_i = \zeta_i'(0)$  of the geodesics  $\zeta_i$  converge as  $i \rightarrow \infty$  to a direction  $v$  with footpoint  $\gamma(0)$ . Let  $\zeta : [0, T) \rightarrow \mathcal{T}(S)$  be the WP-ray with direction  $v$ . Then  $\zeta_i \rightarrow \zeta$  uniformly on compact subsets of  $[0, T)$ .

We claim that the length of  $\nu$  is bounded along  $\zeta$ . Namely, for each  $i$  let  $\mu_i$  be the measured geodesic lamination of  $\gamma_i(0)$ -length one which we obtain from  $\Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  by multiplication with a positive constant. Since  $\gamma_i(0) \rightarrow \gamma(0)$  ( $i \rightarrow \infty$ ) there is a number  $\epsilon > 0$  such that  $\gamma_i(0) \in \mathcal{T}(S)_{\epsilon}$  for all  $i$ . This means that the shortest length of a simple closed curve for the metric  $\gamma_i(0)$  is not smaller than  $\epsilon$ . Therefore  $\mu_i$  is obtained from the counting measure for  $\Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  by multiplication with a constant which does not exceed  $1/\epsilon$ . By definition of the map  $\Upsilon_{\mathcal{T}}$ , the  $\gamma_i(n_i)$ -length of  $\mu_i$  is not bigger than  $\chi_0/\epsilon$ . Thus by convexity, the length of  $\mu_i$  along  $\zeta_i$  is uniformly bounded, independent of  $i$ .

By the above consideration and continuity of the length pairing, as  $i \rightarrow \infty$  the measured geodesic laminations  $\mu_i$  converge weakly to the measured geodesic lamination  $\nu$ . Since  $\zeta_i \rightarrow \zeta$  locally uniformly, continuity of the length pairing implies that the length of  $\nu$  is uniformly bounded along  $\zeta$  (compare the more detailed argument in the proof of Proposition 2.8).

Since  $\nu$  fills  $S$ , the WP-ray  $\zeta$  is infinite. Namely, otherwise there is a simple closed curve  $c$  on  $S$  so that the  $\zeta(t)$ -length of  $c$  tends to zero as  $t \rightarrow T$ . But  $i(c, \nu) > 0$  and consequently in this case the length of  $\nu$  is unbounded along  $\zeta$  which is a contradiction. The lemma is proven.  $\square$

The following angle control is the second and last preparatory step toward the construction of a WP-ray associated to a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  whose initial direction is contained in the preimage of one of the sets  $B(j, \ell)$ .

**Lemma 6.5.** *Let  $B \subset \mathcal{Q}(S)$  be a compact set consisting of quadratic differentials with uniquely ergodic vertical and horizontal measured geodesic laminations. Then there are numbers  $\alpha = \alpha(B) > 0$  and  $R_0 = R_0(B) > 0$  with the following property. Let  $\tilde{B} \subset \tilde{\mathcal{Q}}(S)$  be the preimage of  $B$  and let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be a Teichmüller geodesic with initial velocity  $\gamma'(0) \in \tilde{B}$ . Let  $R_1, R_2 \geq R_0$  and let  $\xi_1, \xi_2$  be the Weil-Petersson*

geodesics which connect  $\gamma(0) = \xi_1(0) = \xi_2(0)$  to  $\gamma(-R_1), \gamma(R_2)$ . Then the angle  $\angle_{\gamma(0)}(\xi'_1(0), \xi'_2(0))$  at  $\gamma(0)$  between the geodesics  $\xi_1, \xi_2$  is at least  $\alpha$ .

*Proof.* We argue by contradiction and we assume that there is a set  $B$  as in a lemma for which the statement of the lemma does not hold. Let  $\tilde{B} \subset \tilde{\mathcal{Q}}(S)$  be the preimage of  $B$ . By assumption, there is a sequence of Teichmüller geodesics  $\gamma_i : \mathbb{R} \rightarrow \mathcal{T}(S)$  with initial velocity  $\gamma'(0) \in \tilde{B}$ , and there is a sequence of numbers  $R_i, T_i \rightarrow \infty$  so that the angle between the WP-geodesics  $\zeta_i, \xi_i$  connecting  $\gamma_i(0)$  to  $\gamma_i(-R_i), \gamma_i(T_i)$  tends to zero as  $i \rightarrow \infty$ .

By invariance under the action of the mapping class group and cocompactness of the action of  $\text{Mod}(S)$  on  $\tilde{B}$ , up to passing to a subsequence we may assume that the Teichmüller geodesics  $\gamma_i$  converge as  $i \rightarrow \infty$  to a Teichmüller geodesic  $\gamma$ . By Lemma 6.4, by passing to another subsequence we may assume that the WP-geodesics  $\zeta_i, \xi_i$  converge as  $i \rightarrow \infty$  to infinite Weil-Petersson rays  $\zeta, \xi : [0, \infty) \rightarrow \mathcal{T}(S)$ .

Let  $q_h, q_v$  be the horizontal and vertical measured geodesic lamination, respectively, of the area one quadratic differential  $q$  which is the unit cotangent vector of  $\gamma$  at  $\gamma(0)$ . By Lemma 6.4, the length of  $q_v$  is bounded along  $\zeta$ , and the length of  $q_h$  is bounded along  $\xi$ .

On the other hand, by assumption, the angle at  $\gamma_i(0)$  between  $\zeta_i, \xi_i$  converges to zero as  $i \rightarrow \infty$  and therefore we have  $\zeta = \xi$ . As a consequence, the lengths of both  $q_h, q_v$  are bounded along  $\zeta$ . But the measured geodesic laminations  $q_h, q_v$  bind  $S$  (i.e. we have  $i(\mu, q_h) + i(\mu, q_v) > 0$  for every measured geodesic lamination  $\mu$  on  $S$ ) and hence the function on  $\mathcal{T}(S)$  which associates to a point  $x \in \mathcal{T}(S)$  the value  $\ell_x(q_h) + \ell_x(q_v)$  is proper (see Theorem 1.2 of [K92]). Since  $\zeta$  is an infinite ray, this is a contradiction. The lemma follows.  $\square$

Denote by  $P : T^*\mathcal{T}(S) \rightarrow \mathcal{T}(S)$  the canonical projection of the vector bundle  $T^*\mathcal{T}(S)$  of all holomorphic quadratic differentials (which is the cotangent bundle of  $\mathcal{T}(S)$ ) onto the base. Then  $P$  restricts to the canonical projection of the sphere bundles for both the Teichmüller metric and the Weil-Petersson metric. The next theorem is the first part of Theorem 1 from the introduction.

**Theorem 6.6.** *There is a measurable conjugacy  $\Lambda : \mathcal{E} \rightarrow \mathcal{Q}_{WP}(S)$  of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $\mathcal{E}$  into the geodesic flow of the Weil-Petersson metric. Its restriction to every compact subset of  $\mathcal{E}$  is continuous.*

*Proof.* Recall that the preimage  $\tilde{\mathcal{E}} \subset \tilde{\mathcal{Q}}(S)$  of  $\mathcal{E}$  consists of differentials with uniquely ergodic horizontal and vertical measured geodesic laminations.

For all  $j, \ell$  let  $\tilde{E}(j, \ell) \subset \tilde{\mathcal{E}}$  be the preimage of  $E(j, \ell)$  in  $\tilde{\mathcal{Q}}(S)$ . Let moreover  $\tilde{B}(j, \ell)$  be the preimage of  $B(j, \ell)$ .

Fix  $(j, \ell)$  and recall that  $E(j, \ell) \subset B(j, \ell)$ . By Lemma 6.2 we may apply Lemma 6.4 and Lemma 6.5 to the sets  $B(j, \ell)$ . Thus for every  $q \in E(j, \ell)$  and every lift  $\tilde{q}$  of  $q$  there are unique WP-geodesics rays  $\zeta_+(\tilde{q}), \zeta_-(\tilde{q})$  which are limits of segments connecting points on the forward or backward geodesic subray of the geodesic  $\gamma_{\tilde{q}}$  with initial velocity  $\tilde{q}$ . The angle between  $\zeta_+(\tilde{q})$  and  $\zeta_-(\tilde{q})$  is bounded from below

by  $\alpha = \alpha(B(j, \ell))$  as in Lemma 6.5. Moreover, these rays depend continuously on  $\tilde{q} \in \tilde{E}(j, \ell)$ .

Let  $\delta = \delta(1/j)$  and  $\eta = \eta(1/j)$  as in Proposition 5.4. Let  $k = k(\delta, \alpha) > 0$  be as in the first part of Lemma 3.1. Let  $n > \max\{\ell, k/\eta\}$  be such that  $\Phi_{\mathcal{T}}^n q \in B(j, \ell)$ ; such a number exists by the definition of  $E(j, \ell)$ . By the choice of  $\alpha$ , if  $n$  is sufficiently large then for sufficiently large  $R \geq k/\eta$  the angle at  $\gamma_{\tilde{q}}(0)$  of the geodesic triangle with vertices  $\gamma_{\tilde{q}}(n)$ ,  $\gamma_{\tilde{q}}(0)$ ,  $\gamma_{\tilde{q}}(-R)$  is at least  $\alpha$  and the same holds true for the angle at  $\gamma_{\tilde{q}}(n)$  of the triangle with vertices  $\gamma_{\tilde{q}}(n+R)$ ,  $\gamma_{\tilde{q}}(n)$ ,  $\gamma_{\tilde{q}}(0)$ . Proposition 5.4 shows moreover that  $\ell_{\delta\text{-thick}}(\gamma_{\tilde{q}}[0, n]) \geq k$ . Thus by Corollary 3.2, there is a unique Weil-Petersson geodesic  $\xi(\tilde{q})$  which is forward asymptotic to  $\zeta_+(\tilde{q})$  and backward asymptotic to  $\xi_-(\tilde{q})$ . Moreover, this geodesic depends continuously on  $\tilde{q} \in \tilde{E}(j, \ell)$ . Its projective ending measures are the classes  $[\tilde{q}^v], [\tilde{q}^h]$  of the vertical and horizontal measured geodesic laminations of  $\tilde{q}$ .

We complete the proof of the theorem using the arguments from the proof of Proposition 2.8. Namely, let  $\tilde{q} \in \tilde{E}(j, \ell)$  and let  $\tilde{q}^v, \tilde{q}^h$  be the vertical and the horizontal measured geodesic lamination of  $\tilde{q}$ , respectively. The function  $x \rightarrow \ell_x(\tilde{q}^v)$  is strictly convex along the Weil-Petersson geodesic  $\xi(\tilde{q})$  and tends to zero as  $t \rightarrow \infty$ . Let  $\tilde{\Lambda}(\tilde{q})$  be the unit cotangent vector of  $\xi(\tilde{q})$  at the unique point  $\xi(\tilde{q})(s)$  where this length equals one. Since length functions are strictly convex along Weil-Petersson geodesics, the assignment  $t \rightarrow \tilde{\Lambda}(\Phi_{\mathcal{T}}^t \tilde{q})$  is a homeomorphism of the orbit of the Teichmüller flow through  $\tilde{q}$  onto the orbit of the Weil-Petersson flow through  $\tilde{\Lambda}(\tilde{q})$ .

Since the Teichmüller geodesic  $t \rightarrow P\Phi_{\mathcal{T}}^t \tilde{q}$  is uniquely determined by the projective classes of the horizontal and vertical measured geodesic laminations, respectively, and these projective measured geodesic laminations are the ending measures of the WP-geodesic determined by  $\tilde{\Lambda}(\tilde{q})$ , the map  $\tilde{q} \in \tilde{E}(j, \ell) \rightarrow \tilde{\Lambda}(\tilde{q})$  is injective, moreover it is continuous and equivariant under the action of the mapping class group. Thus this map projects to a measurable map  $\Lambda : \mathcal{E} \rightarrow \mathcal{Q}_{WP}(S)$  which defines a conjugacy of the Teichmüller geodesic flow on  $\mathcal{E}$  into the Weil-Petersson geodesic flow. Its restriction to each of the sets  $B(j, \ell)$  is continuous.  $\square$

**Remark 6.7.** The proof of Theorem 6.6 moreover shows that for every  $q \in \mathcal{E}$  there is a number  $c = c(q) > 0$  with the following property. Let  $\tilde{\Lambda} : \tilde{\mathcal{Q}}(S) \rightarrow \tilde{\mathcal{Q}}_{WP}(S)$  be a  $\text{Mod}(S)$ -equivariant lift of  $\Lambda$ , defined on the preimage  $\tilde{\mathcal{E}}$  of  $\mathcal{E}$ , and let  $\tilde{q} \in \tilde{\mathcal{E}}$  be a lift of  $q$ . Then there is a sequence  $t_i$  ( $i \in \mathbb{Z}$ ) with  $t_i \rightarrow \pm\infty$  ( $i \rightarrow \pm\infty$ ) and such that  $d_{\mathcal{T}}(P\Phi_{\mathcal{T}}^{t_i} \tilde{q}, P\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i} \tilde{q})) \leq c$  for all  $i$ .

Namely, by continuity, for any compact set  $\tilde{K} \subset \tilde{\mathcal{Q}}(S)$  which projects onto  $B(j, \ell)$  there is some  $c > 0$  such that  $d_{\mathcal{T}}(P\tilde{q}, P\tilde{\Lambda}(\tilde{q})) \leq c$  for every  $\tilde{q} \in \tilde{K}$ . As for every  $q \in E(j, \ell)$  the flow line of  $\Phi_{\mathcal{T}}^t$  through  $q$  intersects  $B(j, \ell)$  for arbitrarily large and small times, by equivariance this number  $c$  satisfies the properties stated in the proposition for all  $q \in E(j, \ell)$ .

We conclude this section with the proof of the first part of Theorem 3 from the introduction (which is a version of a special case of Theorem 6.6). As in the introduction, we always denote by  $J, J'$  a closed connected subset of  $\mathbb{R}$ .



**Proposition 6.8.** (1) *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property. For every Teichmüller geodesic  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  there is a Weil-Petersson geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*

(2) *Let  $K \subset \mathcal{Q}(S)$  be any compact set which is invariant under the Teichmüller geodesic flow  $\Phi_{\mathcal{T}}^t$ . Then there is a conjugacy  $\Lambda : K \rightarrow \mathcal{Q}_{WP}(S)$  of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $K$  into the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric.*

*Proof.* The second part of the proposition is immediate from Theorem 6.6.

The argument for the first part is similar to the proof of Proposition 2.8. Namely, let  $J \subset \mathbb{R}$  be a closed connected set containing 0 and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic. We say that the projective measured geodesic lamination  $[\beta]$  defined by a simple closed curve  $\beta \in \mathcal{C}(S)$  is *realized* at some  $t \in J$  if the  $\gamma(t)$ -length of  $\beta$  does not exceed  $\chi_0$ . If  $J$  contains  $[0, \infty)$  then we say that the projectivization  $[q_v]$  of the vertical measured geodesic lamination defined by the unit cotangent vector  $q$  of  $\gamma$  at  $\gamma(0)$  is realized at the right endpoint of  $J$ , and similarly for a left infinite endpoint.

Let  $\Gamma$  be the set of all triples  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-)$  with the following properties.

- (1)  $J \subset \mathbb{R}$  is a closed connected set containing 0.
- (2)  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is a Teichmüller geodesic.
- (3)  $\lambda_+, \lambda_-$  are measured geodesic laminations of  $\gamma(0)$ -length one, and the projective measured geodesic lamination  $[\lambda_+]$  is realized at the right endpoint of  $J$ , the projective measured geodesic lamination  $[\lambda_-]$  is realized at the left endpoint of  $J$ .

We equip  $\Gamma$  with the product topology, using the weak\*-topology on  $\mathcal{ML}$  for the second and the third component of the triple and the compact-open topology for the arc  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . Note that this topology is metrizable. Moreover,  $\Gamma$  is invariant under the natural action of the mapping class group.

We claim that the action of  $\text{Mod}(S)$  on  $\Gamma$  is cocompact. Since  $\text{Mod}(S)$  acts cocompactly on  $\mathcal{T}(S)_\epsilon$ , for this it is enough to show that the following holds true. If  $\gamma_i : J_i \rightarrow \mathcal{T}(S)_\epsilon$  ( $i > 0$ ) is any sequence of Teichmüller geodesics which converge locally uniformly to a Teichmüller geodesic  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ , if the projective measured geodesic lamination  $[\lambda_i]$  is realized at the right endpoint of  $J_i$  and if  $[\lambda_i] \rightarrow [\lambda]$  in  $\mathcal{PML}$  ( $i \rightarrow \infty$ ) then  $[\lambda]$  is realized at the right endpoint of  $J$ . However, that this holds true was shown in the proof of Proposition 3.4 of [H10a] (compare also the argument in the proof of Proposition 2.8).

To each triple  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  associate a Weil-Petersson geodesic  $\rho(\gamma, \lambda_+, \lambda_-)$  as follows. Assume first that  $J = [-a, b]$  is bounded. Then there is up to parametrization a unique WP-geodesic  $\xi$  connecting  $\gamma(-a)$  to  $\gamma(b)$ . The restriction to  $\xi$  of the function which associates to  $x \in \mathcal{T}(S)$  the sum  $\ell_x(\lambda_+) + \ell_x(\lambda_-)$  is strictly convex and non-constant (unless  $a = b = 0$ ) and hence it assumes a unique minimum along  $\xi$  [W08]. Let  $\rho(\gamma, \lambda_+, \lambda_-)$  be the parametrization of  $\xi$  so that this minimum is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

If  $J$  is one-sided unbounded, say if  $J = [-a, \infty)$  for some  $a \geq 0$ , then there is a unique infinite WP-geodesic ray  $\xi$  issuing from  $\gamma(-a)$  which is asymptotic to  $\eta(\gamma_+)$  where  $\eta(\gamma_+)$  is as in Lemma 6.4. The function  $x \rightarrow \ell_x(\lambda_+) + \ell_x(\lambda_-)$  assumes a unique minimum along  $\xi$ . We define  $\rho(\gamma, \lambda_+, \lambda_-)$  to be the parametrization of  $\xi$  for which this minimum is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

If  $J$  is two-sided infinite then let  $\rho(\gamma, \lambda_+, \lambda_-)$  be the parametrization of the geodesic  $\zeta(\gamma)$  as in the second part of Lemma 6.5 so that the minimum of the function  $x \rightarrow \ell_x(\lambda_+) + \ell_x(\lambda_-)$  along  $\eta(\gamma)$  is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

By Lemma 6.5 and continuity and convexity of length functions, the assignment which associates to  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  the point  $\rho(\gamma, \lambda_+, \lambda_-)(0)$  is continuous, moreover it is equivariant under the action of the mapping class group. Since  $\text{Mod}(S)$  acts cocompactly on  $\Gamma$ , this means that for every  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  the Teichmüller distance between  $\gamma(0)$  and  $\rho(\gamma, \lambda_+, \lambda_-)(0)$  is uniformly bounded. The first part of the proposition now follows as in the proof of Proposition 2.8.  $\square$

## 7. INVARIANT MEASURES FOR THE TEICHMÜLLER FLOW

Denote by  $h(\mu) \geq 0$  the entropy of a  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  (or of a  $\Phi_{\text{WP}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}_{\text{WP}}(S)$ ). We continue to use the assumptions and notations from Section 2-7

**Theorem 7.1.** *The conjugacy  $\Lambda$  induces a continuous injective map*

$$\Theta : \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{\text{WP}}(\mathcal{Q}_{\text{WP}}(S)).$$

Moreover,

$$h(\Theta(\mu)) \geq h(\mu) / \sqrt{2\pi(2g - 2 + m)}$$

for all  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$ .

*Proof.* Since the Teichmüller space  $\mathcal{T}(S)$  is contractible and the Teichmüller metric on  $\mathcal{T}(S)$  is complete, the action of  $\Phi_{\mathcal{T}}^t$  on  $\tilde{\mathcal{Q}}(S)$  is proper. This implies that the space of oriented geodesics  $\mathcal{G}(S)$  for the Teichmüller metric is a locally compact  $\text{Mod}(S)$ -space (which is naturally homeomorphic to a  $\text{Mod}(S)$ -invariant open subset of  $\mathcal{PML} \times \mathcal{PML} - \Delta$  (where  $\Delta$  denotes the diagonal). This set consists of all pairs  $([\mu], [\nu])$  which bind  $S$ .

The measure  $\mu$  induces a locally finite  $\Phi_{\mathcal{T}}^t$ -invariant  $\text{Mod}(S)$ -invariant measure  $\tilde{\mu}$  on  $\tilde{\mathcal{Q}}(S)$ . Via disintegration, the measure  $\tilde{\mu}$  projects to a locally finite  $\text{Mod}(S)$ -invariant measure  $\hat{\mu}$  on  $\mathcal{G}(S)$ . Since  $\mu$  is ergodic under the Teichmüller flow, the measure  $\hat{\mu}$  is ergodic under the action of  $\text{Mod}(S)$ .

Let  $\tilde{\mathcal{E}} \subset \tilde{\mathcal{Q}}(S)$  be the preimage of the set  $\mathcal{E}$  defined in Section 6. Then  $\tilde{\mathcal{E}}$  is invariant under  $\Phi_{\mathcal{T}}^t$  and hence it projects to a  $\text{Mod}(S)$ -invariant Borel subset  $\hat{\mathcal{E}}$  of  $\mathcal{G}(S)$  of full measure for  $\hat{\mu}$ .

Let  $\mathcal{G}_{\text{WP}}(S)$  be the space of biinfinite geodesics for the Weil-Petersson metric. The conjugacy  $\Lambda : \mathcal{E} \rightarrow \mathcal{Q}_{\text{WP}}(S)$  lifts to a  $\text{Mod}(S)$ -equivariant conjugacy  $\hat{\Lambda} : \hat{\mathcal{E}} \rightarrow \mathcal{G}_{\text{WP}}(S)$ . The push-forward  $\hat{\Lambda}_*(\hat{\mu})$  of the measure  $\hat{\mu}$  is a  $\text{Mod}(S)$ -invariant ergodic

measure on  $\mathcal{G}_{WP}(S)$ . Its product with the standard Lebesgue measure on the flow lines of the Weil-Petersson geodesic flow defines a  $\Phi_{WP}^t$ -invariant  $\text{Mod}(S)$ -invariant locally finite Borel measure on  $\tilde{\mathcal{Q}}_{WP}(S)$  which determines a locally finite Borel measure  $\mu_0$  on  $\mathcal{Q}_{WP}(S)$  (we explain in more detail in the sequel that  $\mu_0$  is in fact finite). The measure  $\mu_0$  is ergodic under the action of the Weil-Petersson geodesic flow since the measure  $\hat{\Lambda}_*(\hat{\mu})$  is ergodic under the action of  $\text{Mod}(S)$ .

Let  $b = \sqrt{2\pi(2g-2+m)}$ . We claim that the total mass of the measure  $\mu_0$  on  $\mathcal{Q}_{WP}(S)$  is bounded from above by  $b$ . To see that this is the case, let  $\psi : \mathcal{E} \times \mathbb{R} \rightarrow \mathbb{R}$  be the measurable function defined by the conjugacy. The function  $\psi$  satisfies the *cocycle identity*

$$(15) \quad \psi(x, s+t) = \psi(x, s) + \psi(\Phi_{\mathcal{T}}^s x, t) \quad (x \in \mathcal{E}, s, t \in \mathbb{R}).$$

Moreover, its restriction to  $\mathcal{E} \times [0, \infty)$  is non-negative, and we have  $\psi(x, 0) < \psi(x, s) < \psi(x, t)$  for  $0 < s < t$ .

Let  $q \in \mathcal{E}$  be a typical point for  $\mu$  and let  $\tilde{q}$  be a lift of  $q$  to  $\tilde{\mathcal{Q}}(S)$ . By Remark 6.7 there is a number  $c > 0$  and there is a sequence  $t_i \rightarrow \infty$  such that  $d_{\mathcal{T}}(P\Phi_{\mathcal{T}}^{t_i}\tilde{q}, P\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i}\tilde{q})) \leq c$ . Now  $\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i}\tilde{q}) = \Phi_{WP}^{\psi(q, t_i)}\tilde{\Lambda}(\tilde{q})$  and hence since geodesics for the Weil-Petersson metric are globally length minimizing, Lemma 5.2 shows that  $\psi(q, t_i) \leq b(t_i + 2c)$  for all  $i$ . As a consequence, we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \psi(q, t) \leq b.$$

Since  $q \in \mathcal{E}$  was an arbitrary typical point for  $\mu$ , the Birkhoff ergodic theorem together with the cocycle identity (15) implies that the (non-negative) function  $x \rightarrow \psi(x, 1)$  is integrable with respect to  $\mu$ , and  $\int \psi(x, 1) d\mu \leq b$ .

Recall that length functions are smooth along Weil-Petersson geodesics. Therefore by construction of the conjugacy  $\Lambda$ , for every  $q \in \mathcal{E} \subset \mathcal{Q}(S)$  the function  $t \rightarrow \psi(q, t)$  is continuously differentiable, with derivative  $f(q)$  at  $t = 0$  depending measurably on  $q$ . By invariance of the measure  $\mu$  under the Teichmüller flow we have

$$\int f d\mu = \int \left( \int_0^1 f(\Phi_{\mathcal{T}}^t q) dt \right) d\mu(q) = \int \psi(q, 1) d\mu(q) \leq b.$$

On the other hand, for  $\mu$ -almost every  $q$  the Radon-Nikodym derivative of  $\mu_0$  with respect to  $\Lambda_*(\mu)$  exists at  $\Lambda(q)$  and equals  $f(q)$ . Therefore we have

$$\mu_0(\mathcal{Q}_{WP}(S)) = \int f d\mu \leq b$$

as claimed.

As a consequence, the conjugacy  $\Lambda$  induces a map

$$\Theta : \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{WP}(\mathcal{Q}_{WP}(S)).$$

Namely, we showed so far that  $\Lambda$  defines a map  $\hat{\Theta}$  from the set of ergodic  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure on  $\mathcal{Q}(S)$  to a  $\Phi_{\mathcal{Q}}^t$ -invariant Borel measure on  $\mathcal{Q}_{WP}(S)$  whose total mass is at most  $b$ . The map  $\hat{\Theta}$  does not depend on any choices made and hence it is compatible with convex combinations. Thus it naturally

extends to a map from  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  into the space of  $\Phi_{WP}^t$ -invariant Borel measures on  $\mathcal{Q}_{WP}(S)$  of total mass at most  $b$ . We then define

$$\Theta(\mu) = \hat{\Theta}(\mu)/\hat{\Theta}(\mu)(\mathcal{Q}_{WP}(S)).$$

We claim that the map  $\Theta$  is injective. Namely, by construction, if  $\mu$  is an ergodic  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure on  $\mathcal{Q}(S)$  and if  $\tilde{q} \in \mathcal{Q}^1(S)$  is the lift of a typical point for  $\mu$  with vertical and horizontal measured geodesic laminations  $\tilde{q}^v, \tilde{q}^h$ , respectively, then there is a lift of a typical point for  $\Theta(\mu)$  which determines a biinfinite Weil-Petersson geodesic with forward and backward ending measures  $[\tilde{q}^v], [\tilde{q}^h]$ . This Weil-Petersson geodesic is recurrent and hence by Theorem 1.3 of [BMM10] and the fact that  $\overline{\mathcal{T}}(S)$  is a CAT(0)-space without flat strips contained in its open dense subset  $\mathcal{T}(S)$ , such a recurrent geodesic is determined up to parametrization by its projective ending measures. As the  $\Phi_{\mathcal{T}}^t$ -orbit of  $\tilde{q}$  is determined by the pair  $([\tilde{q}^v], [\tilde{q}^h])$  as well, this means that the pairs of all ending measures of all geodesics whose initial cotangents are typical for  $\Theta(\mu)$  determine both  $\Theta(\mu)$  and  $\mu$ . In other words, the restriction of the map  $\Theta$  to the extreme points of  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  is injective, and its image consists of extreme points of  $\mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$ . By naturality of the map  $\hat{\Theta}$  with respect to convex combination, injectivity of  $\Theta$  follows.

Next we show that  $h(\Theta(\mu)) \geq h(\mu)/b$  for every measure  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$ . For this we use Rudolph's theorem (see Section 11.4 in [CFS82]) and Abramov's formula. Namely, let  $\epsilon > 0$ . Then there is a *special representation* of the flow  $\Phi_{\mathcal{T}}^t$  on  $(\mathcal{E}, \mu)$  given by a Lebesgue space  $(M, \nu)$ , a measure preserving automorphism  $H : (M, \nu) \rightarrow (M, \nu)$  and a roof function  $\rho : M \rightarrow [1 - \epsilon, 1]$ . The flow  $\Phi_{\mathcal{T}}^t$  on  $\mathcal{E}$  is just the vertical flow on the space  $\{(x, t) \in M \times \mathbb{R} \mid 0 \leq t < \rho(x)\} / \sim$  where  $(x, \rho(x)) \sim (Hx, 0)$  for all  $x$ . The measure  $\mu$  is the product of  $\nu$  with the Lebesgue measure on  $\mathbb{R}$ . In particular, since  $\mu$  is a probability measure, the total mass of  $\nu$  is contained in the interval  $[1, \frac{1}{1-\epsilon}]$ .

Let  $h_1 \geq 0$  be the entropy of the  $H$ -invariant measure  $\nu$ . By Abramov's formula, the entropy  $h(\mu)$  of  $\mu$  equals  $h_1 / \int \rho d\nu$ . Since  $\rho$  assumes values in  $[1 - \epsilon, 1]$  we have  $\int \rho d\nu \in [1 - \epsilon, 1]$ . In particular, the entropy of  $\nu$  is within  $h_1\epsilon$  of the entropy  $h(\mu)$  of  $\mu$ .

The space  $(M, \nu)$  can be thought of as a measurable section for the flow  $\Phi_{\mathcal{T}}^t$ . Via the conjugacy  $\Lambda$ , it determines a measurable section for the flow  $\Phi_{WP}^t$ . Let  $\hat{\rho}$  be the corresponding first return time. Using again Abramov's formula, the entropy of  $\Theta(\mu)$  equals  $h_1 / \int \hat{\rho} d\nu$ .

Now for each  $t \in [0, 1 - \epsilon]$  the set  $\Phi_{\mathcal{T}}^t M$  is a measurable section for  $\Phi_{\mathcal{T}}^t$  as well to which the above reasoning can be applied. Since  $t \rightarrow \psi(x, t)$  is increasing and non-negative we can estimate

$$\int \hat{\rho} d\nu \leq \frac{1}{1 - \epsilon} \int_M \int_0^{1-\epsilon} \psi(\Phi_{\mathcal{T}}^t x, 1) dt d\nu \leq \frac{1}{1 - \epsilon} \int \psi(x, 1) d\mu \leq \frac{1}{(1 - \epsilon)} b.$$

But  $\epsilon > 0$  was arbitrary and therefore  $h(\Theta(\mu)) \geq h(\mu)/b$  as claimed.

Finally we are left with showing that the map  $\Theta$  is continuous with respect to the weak\*-topology. For this it suffices to show that this holds true for the map  $\hat{\Theta}$ .

As  $\hat{\Theta}$  is natural with respect to convex combinations, standard properties of the weak\*-topology, for this it is enough to show the following. Let  $(\mu_i) \subset \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  be a sequence of ergodic measures converging to an ergodic measure  $\mu$ . Then  $\hat{\Theta}(\mu_i) \rightarrow \hat{\Theta}(\mu)$ .

To see that this is the case, note first that the locally finite  $\text{Mod}(S)$ -invariant measures  $\hat{\mu}_i$  on  $\mathcal{G}(S)$  which are the disintegrations of the lifts  $\tilde{\mu}_i$  of the measures  $\mu_i$  to  $\tilde{\mathcal{Q}}(S)$  converge weakly to the locally finite  $\text{Mod}(S)$ -invariant measure  $\hat{\mu}$ . Let  $K \subset \mathcal{G}(S)$  be any compact set consisting of typical points for  $\hat{\mu}$  in the above sense. In particular,  $K$  is contained in the support of  $\hat{\mu}$ . Moreover, we may assume that  $\hat{\mu}(K') < \hat{\mu}(K)$  for every proper compact subset  $K'$  of  $K$ .

For  $j > 0$  let  $U_j$  be an open relative compact neighborhood of  $K$  with  $U_j \supset U_{j+1}$  and  $\cap_j U_j = K$ . Then  $\mu(K) = \lim_{j \rightarrow \infty} \mu(U_j)$ . For each  $j$  we have

$$\liminf_{i \rightarrow \infty} \hat{\mu}_i(U_j) \geq \hat{\mu}(K).$$

Moreover, as  $K$  is compact, we also have  $\limsup_{i \rightarrow \infty} \hat{\mu}_i(K) \leq \hat{\mu}(K)$ .

Since the measures  $\hat{\mu}_i$  are Borel regular, for every  $j$  we can find a number  $i(j) > 0$  with  $i(j+1) > i(j)$  and a compact subset  $K_j \subset U_j$  consisting of typical points for  $\hat{\mu}_{i(j)}$  and such that  $\hat{\mu}_{i(j)}(K_j) \geq \hat{\mu}(K)$ . By passing to a subsequence we may assume that the compact sets  $K_{i(j)}$  converge as  $j \rightarrow \infty$  in the Hausdorff topology to a compact set  $C$ . Then  $C \subset \cap_j U_j = K$ , on the other hand also  $\hat{\mu}(C) \geq \hat{\mu}(K)$  and hence  $C = K$ . In particular, for every  $\gamma \in K$  there is a sequence  $\gamma_j \in K_{i(j)}$  with  $\gamma_j \rightarrow \gamma$ .

A point  $\gamma \in K$  is determined by its pair of projective ending measures  $([\xi^+], [\xi^-])$ . If  $\gamma_j \rightarrow \gamma$  then the projective ending measures  $([\xi_j^+], [\xi_j^-])$  of  $\gamma_j$  converge to the projective ending measures of  $\gamma$ . By continuous dependence of Teichmüller geodesic on its pair of ending lamination this implies the following. There is a compact set  $B \subset \tilde{\mathcal{Q}}(S)$  so that the cotangent line of each of the geodesics  $\gamma_j, \gamma$  intersects  $B$ . Moreover, the map  $\tilde{\Lambda}$  is defined on  $B$ .

By Theorem 6.6, the restriction of  $\tilde{\Lambda}$  to every compact subset of  $\tilde{\mathcal{E}}$  is continuous. But this just means that if  $\gamma_j \in K_j$  and if  $\gamma_j \rightarrow \gamma$  then  $\hat{\Lambda}(\gamma_j) \rightarrow \hat{\Lambda}(\gamma)$ . By the above discussion and the definition of the weak\*-topology, we have  $\hat{\Theta}(\mu_j) \rightarrow \hat{\Theta}(\mu)$  which is what we wanted to show.  $\square$

## 8. INVARIANT MEASURES FOR THE WEIL PETERSSON FLOW

The main goal of this section is to show Theorem 2 from the introduction.

The proof relies on estimating the decay of length of an ending measure along an orbit for  $\Phi_{WP}^t$  which is typical for an invariant ergodic Borel probability measure  $\nu$  on  $\mathcal{Q}_{WP}(S)$ .

For a quadratic differential  $\tilde{z} \in \tilde{\mathcal{Q}}_{WP}(S)$  denote by  $\zeta_{\tilde{z}}$  the maximal WP-geodesic with initial velocity  $\tilde{z}$ . Call a point  $q \in \mathcal{Q}_{WP}(S)$  *birecurrent* if it is contained in its own  $\alpha$ - and  $\omega$ -limit set for the action of  $\Phi_{WP}^t$ . Let  $\nu$  be a  $\Phi_{WP}^t$ -invariant

ergodic Borel probability measure on  $\mathcal{Q}_{WP}(S)$ . Then a typical point  $q$  for  $\nu$  is birecurrent. A preimage  $\tilde{q}$  of  $q$  in  $\tilde{\mathcal{Q}}_{WP}(S)$  defines a biinfinite WP-geodesic  $\zeta_{\tilde{q}}$ . This geodesic admits filling topological ending laminations  $\lambda_+(\tilde{q}), \lambda_-(\tilde{q})$ . Every forward (or backward) ending measure, i.e. an ending measure for the ray  $\zeta_{\tilde{q}}[0, \infty)$  (or for the ray  $\zeta_{\tilde{q}}(-\infty, 0]$ ), is supported in  $\lambda_+(\tilde{q})$  (or in  $\lambda_-(\tilde{q})$ ), but recurrence of the orbit does not necessarily imply that such an ending measure is unique up to scale [BMo14].

Our first goal is to establish that for a typical orbit for  $\nu$ , an ending lamination is uniquely ergodic. We begin with a length estimate for measured laminations along a typical orbit for  $\nu$ .

Let as before  $P : T^*\mathcal{T}(S) \rightarrow \mathcal{T}(S)$  be the canonical projection. Let  $z \in \mathcal{Q}_{WP}(S)$  and let  $\tilde{z} \in \tilde{\mathcal{Q}}_{WP}(S)$  be a preimage of  $z$ . For a number  $u > 0$  define a measured lamination  $\beta$  to be *u-admissible* for  $\tilde{z}$  if the length of  $\beta$  is decreasing along the segment  $\zeta_{\tilde{z}}[0, u]$ . This only depends on the projective class of the lamination. Moreover, it is invariant under the action of  $\text{Mod}(S)$  on  $\tilde{\mathcal{Q}}_{WP}(S) \times \mathcal{PML}$ .

For the purpose of the next lemma, note that if  $z \in \tilde{\mathcal{Q}}_{WP}(S)$  is the initial velocity of a biinfinite geodesic then for every  $R > 0$  there is a compact neighborhood  $B$  of  $\tilde{z}$  in  $\tilde{\mathcal{Q}}_{WP}(S)$  so that for every  $y \in B$  the WP-geodesic with initial velocity  $y$  is defined on  $[-R, R]$ .

**Lemma 8.1.** *Let  $q$  be a typical point for  $\nu$ . Then there is a number  $T > 0$ , and there is a compact neighborhood  $V$  of  $q$  with the following properties. Let  $z \in V$  and let  $\beta \in \mathcal{ML}$  be  $T$ -admissible for a preimage  $\tilde{z}$  of  $z$ ; then*

$$\log \ell_\beta(P\tilde{z}) - \log \ell_\beta(P\Phi_{WP}^T(\tilde{z})) \geq 10.$$

*Proof.* Let  $\mu$  be an ending measure for the geodesic  $\zeta_{\tilde{q}}$  with initial velocity a lift  $\tilde{q}$  of  $q$ . Then the length of  $\mu$  is strictly decreasing along  $\zeta_{\tilde{q}}$ . Moreover, if  $\xi$  is a measured lamination whose length strictly decreases along  $\zeta_{\tilde{q}}$  then  $\xi$  belongs to the cone  $\Delta \subset \mathcal{ML}$  of measured laminations whose support coincides with the support of  $\mu$ .

We argue by contradiction and we assume that the lemma does not hold. Then there is a sequence  $t_i \rightarrow \infty$ , a sequence  $\tilde{q}_i \in \tilde{\mathcal{Q}}_{WP}(S)$  with  $\tilde{q}_i \rightarrow \tilde{q}$ , and for each  $i$  there is a  $t_i$ -admissible lamination  $\xi_i \in \mathcal{ML}$  for  $\tilde{q}_i$  so that  $\log \ell_{\xi_i}(P\tilde{q}_i) = 1$  and  $\log \ell_{\xi_i}(P\Phi^{t_i}(\tilde{q}_i)) \geq -10$ . By compactness of  $\mathcal{PML}$  and continuity of length, after passing to a subsequence we may assume that the measured laminations  $\xi_i$  converge as  $i \rightarrow \infty$  to a measured lamination  $\xi$  with  $\ell_\xi(P\tilde{q}) = 1$ .

By continuity of length functions and convexity of length functions along WP-geodesics, the length of  $\xi$  is decreasing along the geodesic  $\zeta_{\tilde{q}}$  (compare the proof of Lemma 2.7 and Proposition 2.8 where such an argument is used for the first time in this work). Thus  $\xi$  is contained in the cone  $\Delta$ , in particular the length of  $\xi$  tends to zero along  $\zeta_{\tilde{q}}$ .

On the other hand, by the definition of admissibility, we have  $\log \ell_{\xi_i}(P\Phi^s(\tilde{q}_i)) \geq -10$  for all  $s \in [0, t_i]$  and hence by continuity,  $\log \ell_\xi(\zeta_{\tilde{q}}(s)) \geq -10$  for all  $s \geq 0$ . This is a contradiction which yields the lemma.  $\square$

For a typical point  $q \in \mathcal{Q}_{WP}(S)$  for  $\nu$  and a preimage  $\tilde{q}$  of  $q$  in  $\tilde{\mathcal{Q}}_{WP}(S)$  let as above  $\lambda_+(\tilde{q})$  be the forward ending lamination of  $\tilde{q}$ . This is a topological lamination which a priori may admit more than one transverse measure up to scale. For  $t > 0$  and a transverse measure  $\mu$  for  $\lambda_+(\tilde{q})$  let

$$\tilde{\alpha}(\tilde{q}, t, \mu) = \log \ell_\mu(P\tilde{q}) - \log \ell_\mu(P\Phi_{WP}^t \tilde{q}).$$

This does not depend on the normalization of  $\mu$ . The thus defined function is invariant under the action of  $\text{Mod}(S)$  on  $\tilde{\mathcal{Q}}_{WP}(S) \times \mathcal{PML}$ . The cocycle equality

$$(16) \quad \tilde{\alpha}(\tilde{q}, s+t, \mu) = \tilde{\alpha}(\tilde{q}, s, \mu) + \tilde{\alpha}(\Phi_{WP}^s \tilde{q}, t, \mu)$$

holds true.

Define

$$\tilde{\alpha}(\tilde{q}, t) = \min\{\tilde{\alpha}(\tilde{q}, t, \mu) \mid \mu\}.$$

The function  $\tilde{\alpha} : \tilde{\mathcal{Q}}_{WP}(S) \times \mathbb{R} \rightarrow \mathbb{R}$  is invariant under the action of  $\text{Mod}(S)$  and hence it descends to a function

$$\alpha : \mathcal{Q}_{WP}(S) \times [0, \infty) \rightarrow [0, \infty).$$

This function is clearly measurable, and equation (16) implies that

$$(17) \quad \alpha(z, s+t) \geq \alpha(z, s) + \alpha(\Phi_{WP}^s z, t).$$

Thus by the subadditive ergodic theorem [Kr85], for  $\nu$ -almost all  $z$  the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \alpha(z, t) \in [0, \infty]$$

exists, and its value  $\sigma$  is independent of  $z$ .

**Lemma 8.2.**  $\sigma > 0$ .

*Proof.* It suffices to assume that  $\sigma < \infty$ . Let  $q \in \mathcal{Q}_{WP}(S)$  be a typical point for  $\nu$  and let  $V$  be a compact neighborhood of  $q$  as in Lemma 8.1. Denote by  $\epsilon > 0$  the  $\nu$ -mass of  $V$ . Let moreover  $T > 0$  be as in Lemma 8.1. Choose  $n > 0$  sufficiently large that the set

$$Z = \{z \mid |\frac{1}{t} \alpha(z, t) - \sigma| \leq \epsilon/4 \text{ for all } t \geq nT\}$$

satisfies  $\nu(Z) \geq 1 - \epsilon/4$ . By the Birkhoff ergodic theorem, we may assume that for  $z \in Z$  and all  $k > n$  we have

$$\frac{1}{k} \sum_{i=0}^{k-1} \chi_V(\Phi_{WP}^{iT} z) \geq 3\epsilon/4$$

where  $\chi_V$  denotes the characteristic function of  $V$ .

Now by the choice of  $Z, T, \epsilon$ , the  $\Phi_{WP}^{iT}$ -orbit ( $i \geq 1$ ) of a point  $z \in Z$  intersects  $V$  in a frequency of at least  $3\epsilon/4$ . Since the function  $\alpha$  is non-negative, equation (17) and the choice of  $V$  implies that the logarithmic length decrease of any ending measure along an orbit segment through a point in  $Z$  is at least  $3\epsilon/4$ . On the other hand, by definition the minimum of this length decrease over all ending measures is  $\epsilon/4$ -close to  $\sigma$  whence the lemma.  $\square$

Lemma 8.2 is used to relate a typical orbit for  $\nu$  to an orbit for the Teichmüller flow  $\Phi_{\mathcal{T}}^t$  on  $\mathcal{Q}(S)$  which recurs to a compact set for arbitrarily large times. The vertical geodesic lamination of a quadratic differential defining such an orbit is known to be uniquely ergodic [M82], and we deduce that the same holds true for the ending lamination of a typical point for  $\nu$ .

Assume for the moment that for  $\nu$ -almost all  $z$  the ending lamination  $\lambda_+(z)$  is uniquely ergodic. For simplicity denote again by  $\lambda_+(z)$  a measured lamination supported in  $\lambda_+(z)$ . We then can unambiguously define

$$\beta(z) = -\frac{d}{dt} \log \ell_{\lambda_+(z)}(\Phi_{WP}^t(z))|_{t=0}.$$

The function  $z \rightarrow \beta(z)$  is measurable and positive. We have

**Proposition 8.3.** *Let  $\nu$  be a  $\Phi_{WP}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}_{WP}(S)$ .*

- (1) *The ending lamination of  $\nu$ -almost every  $q \in \mathcal{Q}_{WP}(S)$  is uniquely ergodic.*
- (2) *There is a  $\Phi_{WP}^t$ -invariant subset  $Z$  of  $\mathcal{Q}_{WP}(S)$  of full measure, and there is a measurable conjugacy  $\Xi: Z \rightarrow (\mathcal{Q}(S), \Phi_{\mathcal{T}}^t)$  into the Teichmüller flow.*
- (3) *There is a measure  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  with  $\Theta(\mu) = \nu$  if and only if  $\int \beta d\nu < \infty$ .*

*Proof.* The set of all projective transverse measures for a forward ending lamination of a typical point for  $\nu$  is a simplex whose dimension is at most  $3g - 3 + m$ . Its vertices are precisely the ergodic projective transverse measures for the lamination. (This is well known, but we were not able to locate the statement in this form in the literature. The work [K73] shows that there are only finitely many ergodic projective invariant measures for an interval exchange transformation, and this implies finiteness of ergodic projective transverse measures for an orientable geodesic lamination which is all we need in the sequel. The case of a non-orientable measured laminations follows via passing to the orientation cover).

By ergodicity of the measure  $\nu$ , the dimension of this simplex of projective transverse measures is  $\nu$ -almost everywhere constant. Similarly, for  $\nu$ -almost every  $z$  the backward ending lamination of  $z$  supports a simplex of transverse measures whose dimension does not depend on  $z$ . We have to show that the dimension of these simplices equals zero almost everywhere.

Let  $n_+ \geq 1, n_- \geq 1$  be the number of vertices of the forward and backward simplex, respectively, and let  $Z$  be a  $\Phi_{WP}^t$ -invariant Borel set of full measure consisting of points where these simplices are defined. There is an  $n = n_+ \cdot n_-$ -sheeted cover  $Z_n$  of  $Z$  as follows. Each point in  $Z_n$  corresponds to a triple  $(q, [\xi_+], [\xi_-])$  where  $q \in Z$  and where  $[\xi_+]$  (or  $[\xi_-]$ ) is a vertex of the simplex of projective transverse measures for the forward (or backward) ending lamination of  $q$ . The flow  $\Phi_{WP}^t$  on  $Z$  naturally lifts to a flow on  $Z_n$ . This flow preserves a finite Borel measure  $\hat{\nu}$  which projects to  $\nu$ . The measure  $\hat{\nu}$  has at most  $n = n_+ \cdot n_-$  ergodic components.

The preimage  $\tilde{Z}$  of  $Z$  in  $\tilde{\mathcal{Q}}_{WP}(S)$  is a  $\text{Mod}(S)$ -invariant  $\Phi_{WP}^t$ -invariant Borel subset of  $\tilde{\mathcal{Q}}_{WP}(S)$ . The covering  $Z_n$  of  $Z$  induces a (formal)  $n_+ \cdot n_-$ -sheeted covering  $\tilde{Z}_n$  of  $\tilde{Z}$ . A point  $\tilde{z} \in \tilde{Z}$  is a triple  $(\tilde{q}, [\xi_+], [\xi_-])$  which consists of a quadratic



differential  $\tilde{q} \in \tilde{\mathcal{Q}}_{WP}(S)$  and a choice of a pair  $([\xi_+], [\xi_-])$  of ergodic forward and backward projective ending measures for the WP-geodesic determined by  $\tilde{q}$ . The support of each of these measures fills  $S$ , and as in the proof of Lemma 6.5, the supports of  $[\xi_+], [\xi_-]$  are distinct. Thus this pair of projective measured geodesic laminations binds  $S$  and hence it determines a Teichmüller geodesic  $\Psi([\xi_+], [\xi_-])$ .

Let  $\xi_+(\tilde{z}), \xi_-(\tilde{z}) \in \mathcal{ML}$  be the measured laminations whose projective classes are determined by the triple  $\tilde{z}$  and whose lengths on the surface  $P\tilde{q}$  underlying the quadratic differential  $\tilde{z}$  equal one. Write  $i(\xi_+(\tilde{z}), \xi_-(\tilde{z})) = a^{-2}$  where as before,  $i$  is the intersection form. Let  $\tilde{\Xi}(\tilde{z}, [\xi_+], [\xi_-])$  be the unit cotangent vector for the geodesic  $\Psi([\xi_+], [\xi_-])$  with the property that the vertical and horizontal measured geodesic laminations of  $\tilde{\Xi}(\tilde{z}, [\xi_+], [\xi_-])$  equal  $a\xi_+(\tilde{z}), a\xi_-(\tilde{z})$ .

By smoothness and strict convexity of length functions along Weil-Petersson geodesics and by Lemma 8.2, the map  $t \rightarrow \tilde{\Xi}(\Phi_{WP}^t(\tilde{q}), [\xi_+], [\xi_-])$  is a homeomorphism onto the cotangent line of the geodesic  $\Psi([\xi_+], [\xi_-])$ . This construction defines a measurable map  $\tilde{\Xi} : \tilde{Z}_n \rightarrow \tilde{\mathcal{Q}}(S)$  which is equivariant under the action of the mapping class group. Thus this map descends to a measurable map  $\Xi : Z_n \rightarrow \mathcal{Q}(S)$  which conjugates the flow on  $Z_n$  into the Teichmüller flow.

Let  $K \subset Z_n$  be a compact set of positive  $\hat{\nu}$ -measure so that the restriction of  $\Xi$  to  $K$  is continuous. The image of  $K$  under the map  $\Xi$  is compact. The projection of  $K$  to  $Z$  is a compact subset  $K_0$  of  $Z$  of positive measure. By ergodicity of  $\nu$ , for  $\nu$ -almost every  $q \in K_0$  the  $\Phi_{WP}^t$ -orbit of  $q$  recurs to  $K_0$  for arbitrarily large times. As every point in  $K_0$  has  $n$  preimages in  $Z_n$ , this means that there is a Borel subset  $A$  of  $K$  with  $\hat{\nu}(A) > 0$  which consists of points whose orbit under the flow on  $Z_n$  recurs to  $K$  for arbitrarily large times.

For  $z \in A$  the orbit of  $\Xi(z)$  under the Teichmüller flow  $\Phi_{\mathcal{T}}^t$  recurs to the compact set  $\Xi(K)$  for arbitrarily large times. Therefore by Masur's result [M82], for all  $z \in A$  the vertical measured lamination of  $\Xi(z)$  is uniquely ergodic. But this vertical measured lamination is the forward ending measure of the geodesic defined by a lift of  $z$ . Thus for all  $z \in A$  the forward ending lamination for  $q$  is uniquely ergodic. But  $A$  projects to a subset of  $Z$  of positive measure and hence by ergodicity of  $\nu$ , for almost all  $z \in \mathcal{Q}_{WP}(S)$  the forward ending measure is uniquely ergodic. The same argument applies to the backward ending measure. Together the first part of the proposition follows, and the second part is an immediate consequence of the first and the above construction.

The push-forward  $\Xi_*\nu$  of  $\nu$  under the map  $\Xi$  is a Borel probability measure on  $\mathcal{Q}(S)$  which is quasi-invariant under the flow  $\Phi_{\mathcal{T}}^t$ . To construct an invariant measure for  $\Phi_{\mathcal{T}}^t$  we use again a special representation of the flow  $\Phi_{WP}^t$  on  $Z$  given by a Lebesgue space  $(M, \chi)$ , a measure preserving automorphism  $H : (M, \chi) \rightarrow (M, \chi)$  and a roof function  $\rho : M \rightarrow [1 - \epsilon, 1]$ . The flow  $\Phi_{WP}^t$  is just the vertical flow on the space  $\{(x, t) \in M \times \mathbb{R} \mid 0 \leq t < \rho(x)\} / \sim$  where  $(x, \rho(x)) \sim (Hx, 0)$  for all  $x$ . Viewing  $M$  as a Borel section for the flow  $\Phi_{WP}^t$ , we can map  $M$  with  $\Xi$  to a Borel section for  $\Phi_{\mathcal{T}}^t$ . The image  $\Xi_*(\chi)$  is a finite Borel measure which determines a  $\Phi_{\mathcal{T}}^t$ -invariant locally finite Borel measure  $\mu$  on  $\mathcal{Q}(S)$ .

For  $x \in M$  let  $f(x)$  be the length of the orbit segment  $\cup_{0 \leq t < \rho(x)} \Xi(\Phi_{WP}^t(x))$ . It follows as in the proof of Theorem 7.1 that the measure  $\mu$  is finite if and only if  $\int f d\chi < \infty$ . Moreover, if this is the case then  $\Theta(\mu) = \nu$  is immediate from our construction.

We are left with showing that  $\int f d\chi < \infty$  if and only if  $\int \beta d\nu < \infty$ . To this end observe that the restriction of the map  $\Xi$  to a flow line of the Weil-Petersson flow is smooth. Namely, lengths functions are smooth along Weil-Petersson geodesics and strictly convex, and by Lemma 8.2, the length of the forward ending measure decays with exponential rate. Thus if  $\omega(z, t)$  is the function defining the conjugacy  $\Xi$ , i.e. if we have

$$\Xi(\Phi_{WP}^t(z)) = \Phi_{\mathcal{T}}^{\omega(z, t)} \Xi(z),$$

then  $\omega$  can be differentiated in direction of the real parameter. Now using again the explicit construction, the function  $\beta$  in part (3) of the proposition coincides with the function

$$\frac{d}{dt}(t \rightarrow \omega(z, t))|_{t=0} > 0$$

almost everywhere. Thus the measure  $\mu$  is finite if and only if the function  $\beta$  is integrable. The proposition follows.  $\square$

We conclude this section with showing that Proposition 8.3 is not a redundant.

**Proposition 8.4.** *The map  $\Theta$  is not surjective.*

*Proof.* It suffices to construct a (not necessarily ergodic) Borel probability measure  $\nu$  such that  $\int \beta d\nu = \infty$ .

For this let  $\varphi$  be a pseudo-Anosov mapping class which admits an invariant train track  $\tau$  with the following properties. First we require that the transition matrix for the carrying relation  $\varphi(\tau) \prec \tau$  is positive. Second we require that there is a simple closed curve  $c$  smoothly embedded in  $\tau$  as a subgraph consisting of two branches, one large branch  $b$  and one small branch.

Splitting  $\varphi(\tau)$  at  $\varphi(b)$  results in a train track which is obtained from  $\varphi(\tau)$  by a single Dehn twist  $T(\varphi(c))$  about  $\varphi(c)$ . The train track  $T(\varphi(c))(\varphi(\tau))$  is carried by  $\varphi(\tau)$  and hence by  $\tau$ , and the transition matrix  $T(\varphi(c))(\varphi(\tau)) \prec \tau$  is positive.

As a consequence, the mapping class  $T(\varphi(c)) \circ \varphi$  is pseudo-Anosov and admits  $\tau$  as an invariant train track. Iteration of this construction shows that for each  $k > 0$  the mapping class  $T(\varphi(c))^k \circ \varphi$  is pseudo-Anosov. Moreover, for a suitable choice of  $\varphi$ , the closed orbit for the Teichmüller flow in  $\mathcal{Q}(S)$  which defines the conjugacy class of  $T(\varphi(c))^k \varphi$  contains a subarc of fixed positive length in a fixed compact subset  $K$  of  $\mathcal{Q}(S)$ .

Let  $\ell(k)$  be the length of the periodic orbit for the Teichmüller flow which defines the conjugacy class of  $T(\varphi(c))^k \circ \varphi$ . Then  $\ell(k) \rightarrow \infty$  ( $k \rightarrow \infty$ ).

As Teichmüller space with the Weil-Petersson metric is quasi-isometric to the pants graph [B03], the periodic orbits for  $\Phi_{WP}^t$  corresponding to the conjugacy

classes of  $T(\varphi(c))^k \circ \varphi$  have uniformly bounded length, say their length is bounded by  $n > 0$ .

Choose a sequence  $k(i) \rightarrow \infty$  so that  $\sum_i \frac{1}{i^2} \ell(k(i)) = \infty$ . Let  $\mu(i)$  be the (unnormalized)  $\Phi_{WP}^t$ -invariant Borel measure supported on the periodic orbit in  $\mathcal{Q}_{WP}(S)$  which defines the conjugacy class of  $T(\varphi(c))^k \circ \varphi$ . Define a  $\Phi_{WP}^t$ -invariant Borel measure  $\nu$  on  $\mathcal{Q}_{WP}(S)$  by

$$\nu = \sum_i \frac{1}{i^2} \mu(k(i)).$$

This measure is finite and can be normalized to a probability measure.

The conjugacy  $\Xi$  constructed in the proof of Proposition 8.3 maps the measure  $\nu$  to a weighted sum of invariant measures on the periodic orbits for  $\Phi_{\mathcal{T}}^t$ . As  $\sum_i \frac{1}{i^2} \ell(k(i)) = \infty$ , this measure on  $\mathcal{Q}(S)$  is infinite. Thus  $\Theta$  is not surjective.  $\square$

**Remark:** As the space  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  is not compact, Proposition 8.4 does not imply that there is an ergodic Borel probability measure for  $\Phi_{WP}^t$  not contained in the image of  $\Theta$ . We do not know whether such a measure exists.

## 9. THE LEBESGUE LIOUVILLE MEASURE

Recall from Section 7 the definition of the map

$$\Theta : \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{WP}(\mathcal{Q}_{WP}(S)).$$

The goal of this section is to show

**Proposition 9.1.** *The Lebesgue Liouville measure of the Weil-Petersson metric is contained in the image of the map  $\Theta$ .*

*Proof.* Let  $\nu$  be the Lebesgue Liouville measure of the Weil-Petersson metric. Recall from Section 8 the definition of the function  $\beta$ , defined on a Borel subset of  $\mathcal{Q}_{WP}(S)$  which is of full measure for every  $\Phi_{WP}^t$ -invariant Borel probability measure. By the third part of Proposition 8.3, we have to show that  $\int \beta d\nu < \infty$ .

For  $\tilde{q} \in \tilde{\mathcal{Q}}_{WP}(S)$  let  $\zeta_{\tilde{q}}$  be the WP-geodesic with initial velocity  $\tilde{q}$ . Recall that for every  $\tilde{q} \in \tilde{\mathcal{Q}}_{WP}(S)$  and every measured lamination  $\sigma \in \mathcal{ML}$  the derivative

$$\frac{d}{dt} \log \ell_{\sigma}(\zeta_{\tilde{q}}(t))|_{t=0}$$

is defined and depends continuously on  $\tilde{q}$  and  $\sigma$ . Moreover, this derivative does not depend on the normalization of  $\sigma$  and hence this defines a continuous function on  $\tilde{\mathcal{Q}}_{WP}(S) \times \mathcal{PML}$  which is invariant under the action of the mapping class group.

The function  $\tilde{f} : \tilde{\mathcal{Q}}_{WP}(S) \rightarrow (0, \infty)$  defined by

$$\tilde{f}(\tilde{q}) = \max \left\{ \frac{d}{dt} \log \ell_{\sigma}(\zeta_{\tilde{q}}(t))|_{t=0} \mid \sigma \in \mathcal{ML} \right\}$$

is  $\text{Mod}(S)$ -invariant and continuous and hence it descends to a continuous function  $f$  on  $\mathcal{Q}_{WP}(S)$ . By definition of the function  $\beta$ , if  $\mu$  is a  $\Phi_{WP}^t$ -invariant Borel probability measure on  $\mathcal{Q}_{WP}(S)$  with  $\int f d\mu < \infty$  then  $\int \beta d\mu < \infty$ .

The WP-metric on moduli space  $\mathcal{M}(S) = \mathcal{T}(S)/\text{Mod}(S)$  is incomplete. Its completion coincides with the Deligne Mumford compactification  $\overline{\mathcal{M}}(S)$  of  $\mathcal{M}(S)$ . Let

$$\rho : \mathcal{M}(S) \rightarrow (0, \infty)$$

be the function which associates to a point  $x$  the distance from the boundary  $\partial\mathcal{M}(S) = \overline{\mathcal{M}}(S) - \mathcal{M}(S)$ . This boundary consists of surfaces with nodes, i.e. surfaces where each component of some non-trivial simple multicurve has been pinched to a point.

The boundary  $\partial\mathcal{M}(S)$  of  $\mathcal{M}(S)$  is divided into strata according to the number and types of nodes. For a stratum  $\Sigma$  defined by the vanishing of the geodesic-length sum  $\ell = \ell_1 + \dots + \ell_n$ , the distance to the stratum is given locally as

$$d_{WP}(p, \Sigma) = (2\pi\ell)^{1/2} + O(\ell^2)$$

(Corollary 21 of [W03]). In particular, the distance of a point  $x \in \mathcal{M}(S)$  to the boundary  $\partial\mathcal{M}(S)$  equals  $(2\pi\ell_\alpha)^{1/2} + O(\ell_\alpha^2)$  where  $\alpha$  is a systole of  $x$ .

The WP-gradients of the geodesic-length functions also have general expansions [W87]. For a curve  $\alpha$  of length at most  $\epsilon$  we have

$$\|\text{grad}\ell_\alpha\| = \sqrt{\frac{2}{\pi}}\ell_\alpha^{1/2} + O(\ell_\alpha^{3/2})$$

and hence

$$\|d\log\ell_\alpha\| \asymp \ell_\alpha^{-1/2}.$$

where as before, the symbol  $\asymp$  relating two positive functions means that their quotient is bounded from above and below by a universal positive constant. The sharpest infinitesimal length decrease of any normalized measured lamination along any Teichmüller geodesic issuing from  $x$  is the length decrease of a systole along a shortest path connecting  $x$  to  $\partial\mathcal{M}(S)$  [W87, W08].

Let  $P : \mathcal{Q}_{WP}(S) \rightarrow \mathcal{M}(S)$  be the canonical projection. For sufficiently small  $\rho(Pq)$  and a systole  $\alpha$  of  $Pq$  we obtain

$$(18) \quad f(q) \asymp \|d\log\ell_\alpha(Pq)\| \asymp \rho^{-1}(Pq).$$

Thus to show that the function  $f$  is integrable with respect to  $\nu$  it suffices to show that there is a number  $\delta > 0$  such that for sufficiently small  $r$  the WP-volume of the set  $\rho^{-1}(0, r)$  is at most  $r^{1+\delta}$  (compare also [BMW12]).

That this volume is  $O(r^4)$  is immediate from Wolpert's asymptotic expansion of the Weil-Petersson metric near  $\partial\mathcal{M}(S)$  (as explained on p.889 of [BMW12]). Alternatively, we can use Wolpert's formula for the Weil-Petersson Kähler form  $\omega$  in Fenchel Nielsen coordinates for a Bers decomposition by simple closed curves  $\alpha_i$ . This expression equals

$$\omega = \sum_i d\ell_{\alpha_i} \wedge d\theta_{\alpha_i}.$$

The Fenchel Nielsen twist  $\theta_{\alpha_i}$  is the unit speed twist along the simple closed curve  $\alpha_i$ , and its period equals  $\ell_{\alpha_i}$ . Replacing the length-twist coordinates about a systole  $\alpha$  by distance-angle coordinates  $(\rho, \kappa)$  ( $\rho \asymp \ell_\alpha^{-1}, \kappa \in [0, 2\pi)$ ) multiplies the

twist speed by  $2\pi$  times the length of  $\alpha$  which is  $O(\rho^2)$ . Thus in distance-angle coordinates, the volume form is bounded from above by a constant multiple of  $\rho^3$ . This gives

$$\text{vol}\{\rho \leq r\} \asymp \int_0^r r^3 ds = \frac{1}{4}r^4.$$

Together with equation (18), we conclude that the function  $f$  is integrable with respect to the Lebesgue Liouville measure  $\nu$ . This completes the proof of the proposition.  $\square$

To summarize, there is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  such that  $\Theta(\mu) = \nu$ . We do not have a description of the measure  $\mu$ , but we conjecture that in the case of the once punctured torus, this measure is the Lebesgue measure on  $\mathcal{Q}(S)$  (this makes sense even though we assumed throughout the paper that the surface  $S$  is non-exceptional).

We conclude this work with some remarks on absolute continuity and invariant measure classes on  $\mathcal{PML}$ .

The mapping class group  $\text{Mod}(S)$  acts diagonally on  $\mathcal{PML} \times \mathcal{PML} - \Delta$ . The space of oriented geodesic  $\mathcal{G}(S)$  for the Teichmüller metric is the invariant subset of  $\mathcal{PML} \times \mathcal{PML} - \Delta$  of all pairs  $(\mu, \nu)$  which bind  $S$ . Any  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  lifts to a measure on  $\tilde{\mathcal{Q}}(S)$  which disintegrates to a  $\text{Mod}(S)$ -invariant locally finite Borel measure  $\hat{\mu}$  on  $\mathcal{G}(S)$ . By Masur's result [M82], the measure  $\hat{\mu}$  gives full mass to the set of pairs of uniquely ergodic projective measured laminations.

The space  $\mathcal{G}_{WP}(S)$  of biinfinite oriented geodesics for the Weil-Petersson metric does not have such an easy description. However, the main result of this paper shows that there is such a description for a  $\text{Mod}(S)$ -invariant subset of  $\mathcal{G}_{WP}(S)$  whose unit tangent lines project to a subset of  $\mathcal{Q}_{WP}(S)$  of full mass for every  $\Phi_{WP}^t$ -invariant Borel probability measure.

To be more precise, call a point  $q \in \mathcal{Q}_{WP}(S)$  *typical* if  $q$  has the following two properties.

- Let  $\tilde{q} \in \tilde{\mathcal{Q}}(S)$  be a preimage of  $q$ . Then  $\tilde{q}$  determines a biinfinite geodesic whose ending measures are uniquely ergodic.
- $q$  is contained in its own  $\alpha$ - and  $\omega$  limit set.

The following is immediate from Proposition 8.3 and the Poincaré recurrence theorem.

**Lemma 9.2.** *The set  $\mathcal{Z} \subset \mathcal{Q}_{WP}(S)$  of typical points has full measure with respect to every invariant Borel probability measure.*

By Lemma 9.2, a  $\Phi_{WP}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}_{WP}(S)$  determines a locally finite  $\text{Mod}(S)$ -invariant Borel measure  $\hat{\mu}$  on  $\mathcal{PML} \times \mathcal{PML} - \Delta$  which gives full mass to the set of pairs of uniquely ergodic projective measured laminations. The measure  $\mu$  is ergodic if and only if  $\hat{\mu}$  is ergodic under the action of  $\text{Mod}(S)$ .

Call an invariant Borel probability measure  $\mu$  on  $\mathcal{Q}_{WP}(S)$  *absolutely continuous with respect to the stable foliation* if the following holds true. Let  $\hat{\mu}$  be the induced invariant measure on  $\mathcal{PML} \times \mathcal{PML} - \Delta$ . Let  $\hat{\mu}_1$  be a conditional measure on a leaf  $\mathcal{PML} \times \{[\beta]\}$  of the product foliation; then for a Borel set  $A \subset \mathcal{PML} \times \{[\beta]\}$  we have  $\hat{\mu}_1(A) = 0$  if and only if  $\hat{\mu}(A \times \mathcal{PML}) = 0$ . Similarly there is a notion of absolute continuity with respect to the unstable foliation.

The classical Hopf argument (as explained in Section III.3 of [Mn87]) implies

**Proposition 9.3.** *If  $\mu$  is absolutely continuous with respect to the stable and unstable foliation then  $\mu$  is ergodic.*

Examples of absolutely continuous measures are the invariant Lebesgue measure for the Teichmüller flow [M82, V86] and the Lebesgue Liouville measure for the Weil-Petersson flow [BMW12].

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